

Lagrangianité de cycles associés à un \mathcal{D} -module holonôme

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Introduction

Cette thèse traite d'algèbre et de géométrie algébrique, plus précisément de théorie algébrique des \mathcal{D} -modules, avec une forte présence de la caractéristique positive. On sait le rôle joué par la variété caractéristique en théorie des \mathcal{D} -modules, nous en étudions ici un analogue lié aux p -courbures. Commençons par quelques rappels sur l'anneau des opérateurs différentiels.

L'anneau des opérateurs différentiels. Pour un morphisme lisse de schémas $X \rightarrow S$, on entend, dans ce travail, par anneau des opérateurs différentiels, le faisceau d'anneaux $D_{X/S}$ sur X engendré par l'anneau des fonctions \mathcal{O}_X et les champs de vecteurs relatifs $\mathcal{T}_{X/S}$, imposant que le commutateur d'un champ de vecteurs et d'une fonction soit la dérivée de celle-ci suivant celui-là et que le commutateur de deux champs de vecteurs soit leur crochet de Lie. Notons que cet anneau est noethérien et que si S est le spectre d'un corps de caractéristique nulle, on retrouve l'anneau des opérateurs différentiels algébriques usuels. Nos considérations ont pour point de départ les propriétés remarquables de $D_{X/S}$ lorsque S est le spectre d'un corps de caractéristique positive p . En effet, d'après [8], $D_{X/S}$ est alors fini sur son centre $Z(D_{X/S})$ et le spectre de ce dernier s'identifie au fibré cotangent $T^*(X/S)'$ de X/S (tordu par le Frobenius, noté ' ici et plus bas). De plus cette identification se fait via la p -courbure

$$\psi_{\nabla} : \partial \mapsto (\nabla(\partial))^p - \nabla(\partial^{[p]}),$$

notion typique à la caractéristique positive, rendant compte du défaut de compatibilité d'une connexion intégrable ∇ à la puissance p -ième $\partial^{[p]}$ des champs de vecteurs ∂ . Enfin, il est également observé dans [8], et c'est une condition technique essentielle pour la suite, que $D_{X/S}$ est une algèbre d'Azumaya sur le cotangent $T^*(X/S)'$.

Explicitons le cas de l'espace affine \mathbb{A}_S^n , S affine d'anneau R . L'anneau des opérateurs différentiels est alors la faisceau associé à l'algèbre de Weyl d'indice n , $A_n(R)$, c'est-à-dire le produit tensoriel sur R

$$\otimes_{i=1}^{i=n} (R \langle x_i, \partial_i \rangle / (\partial_i x_i - x_i \partial_i = 1)).$$

On constate bien que si R est un corps de caractéristique positive p , $A_n(R)$ est finie sur son centre, qui s'identifie à l'algèbre de polynômes $\otimes_{i=1}^{i=n} R[x_i^p, \partial_i^p]$.

L'opérateur de p -courbure est ici celui qui au champ de vecteur ∂_i correspondant à la i -ième coordonnée, associe l'élément central $\partial_i^p - \partial_i^{[p]} = \partial_i^p$ de $A_n(R)$. (La puissance p -ième $\partial_i^{[p]}$ du champ de vecteur particulier ∂_i s'annule.)

Les p -supports. Supposons que S soit un schéma affine d'anneau R , une algèbre de type fini sur \mathbb{Z} , intègre et de corps de fractions de caractéristique nulle. Donc telle que pour chaque idéal maximal \mathfrak{m} de R , le corps $k(\mathfrak{m}) := R/\mathfrak{m}$ soit de caractéristique positive. Pour un S -schéma lisse X , donnons-nous un module M de type fini sur $D_{X/S}$, noté aussi $D_{X/R}$. Sa réduction $k(\mathfrak{m}) \otimes_R M$, pour un idéal maximal \mathfrak{m} de R , est alors un module de type fini sur $D_{X_{\mathfrak{m}}/k(\mathfrak{m})}$ et en particulier, par les rappels ci-dessus, un module de type fini sur le centre $Z(D_{X_{\mathfrak{m}}/k(\mathfrak{m})})$ de $D_{X_{\mathfrak{m}}/k(\mathfrak{m})}$, ce dernier s'identifiant à l'anneau des fonctions sur le fibré cotangent $T^*(X_{\mathfrak{m}}/k(\mathfrak{m}))'$. On définit le p -support de M en \mathfrak{m} comme la sous-variété de $T^*(X_{\mathfrak{m}}/k(\mathfrak{m}))'$, support de $k(\mathfrak{m}) \otimes_R M$ vu comme module sur $Z(D_{X_{\mathfrak{m}}/k(\mathfrak{m})})$. L'objet central de cette thèse, ainsi que l'analogue promis de la variété caractéristique, est la collection des p -supports de M , pour l'ensemble des idéaux maximaux de R . On se permet en fait d'identifier les collections provenant de M et de ses localisés $M[1/r]$ par des éléments non-nuls r de R , ce qu'on exprime parfois en disant "la collection des p -supports pour p suffisamment grand".

Remarquons enfin qu'on construit des modules comme ci-dessus en épaississant (ou déformant) les \mathcal{D} -modules de type fini, où \mathcal{D} est l'anneau des opérateurs différentiels sur une variété lisse sur k , un corps de caractéristique nulle. En effet, l'anneau \mathcal{D} étant noethérien, tout module de type fini M_0 est de présentation finie et peut donc s'épaissir en un module M de type fini sur $D_{X/R}$ tel que $M_0 \cong k \otimes_R M$, pour X lisse sur R comme ci-dessus, R étant par exemple le sous-anneau de k engendré par les coefficients des relations d'une présentation finie de M_0 .

Résultats. Rappelons que le fibré cotangent d'une variété équidimensionnelle lisse X sur un corps est canoniquement muni d'une forme symplectique et que ses sous-variétés lagrangiennes sont celles qui sont de même dimension que X en chacun de leurs points et sur un ouvert dense desquelles la forme symplectique s'annule. Rappelons aussi que l'holonômie est une condition de finitude fondamentale sur les \mathcal{D} -modules, satisfaite par exemple par tous les modules provenant des fibrés à connexion intégrable. Notre résultat principal est le

Théorème 1 (cf. le Théorème 2.2.1)

Soit X un schéma lisse purement de dimension relative n sur S affine d'anneau R comme ci-dessus et soit M un module à gauche de type fini sur $D_{X/R}$. Supposons que $k \otimes_R M$ soit un \mathcal{D} -module holonôme, pour k le corps des fractions de R (k est donc de caractéristique nulle). Alors les p -supports de M sont des sous-variétés lagrangiennes, "pour p suffisamment grand" (i.e. il existe un élément non-nul r de R tel que les p -supports de $M[1/r]$ soient des sous-variétés lagrangiennes).

Pour l'obtenir, on démontre également l'assertion qui suit. Rappelons cette fois-ci qu'une algèbre d'Azumaya sur une variété est scindée si elle est isomorphe à l'algèbre des endomorphismes d'un fibré vectoriel.

Théorème 2 (cf. le Théorème 6.1.4)

Dans la situation du théorème 1 et sous les mêmes hypothèses, l'algèbre d'Azumaya des opérateurs différentiels se scinde sur le lieu régulier du p -support, "pour p suffisamment grand" (i.e. il existe un élément non-nul r de R tel que pour tout idéal maximal \mathfrak{m} de $R[1/r]$, l'algèbre d'Azumaya $D_{X_{\mathfrak{m}}/k(\mathfrak{m})}$ se scinde sur le lieu régulier du p -support de $k(\mathfrak{m}) \otimes_R M$).

L'énoncé du théorème 1 s'inspire du théorème classique d'intégrabilité de la variété caractéristique [19]. Il ne semble toutefois pas se déduire aisément de son modèle. Qui plus est, les objets sur lesquels ces deux assertions portent ont des différences notables, comme il se voit sur les exemples. Mettons en évidence que les p -supports peuvent dépendre non-trivialement de p et qu'ils ne sont en général pas des sous-variétés coniques du cotangent, c'est-à-dire que contrairement à la variété caractéristique, ils ne sont pas nécessairement préservés par l'action du groupe multiplicatif dans les fibres.

Exemples.

- Soient $R = \mathbb{Z}$, $X = \mathbb{A}_{\mathbb{Z}}^n$ et soit M le $D_{X/R}$ -module à gauche de type fini correspondant à la R -connexion intégrable

$$\nabla = d + dg$$

sur \mathcal{O}_X , où g est une section globale de \mathcal{O}_X . De l'identité $(\partial_i + \partial g / \partial x_i)^p = (\partial_i)^p + (\partial g / \partial x_i)^p$ dans $A_n(\mathbb{Z}/p\mathbb{Z})$ [31, 5.2.4], on déduit que le p -support de $\mathbb{Z}/p\mathbb{Z} \otimes_{\mathbb{Z}} M \subset T^* \mathbb{A}_{\mathbb{Z}/p\mathbb{Z}}^n = T^* \mathbb{A}_{\mathbb{Z}/p\mathbb{Z}}^n$ est la réduction modulo p du graphe de $dg \subset T^* \mathbb{A}_{\mathbb{Z}}^n$. Les p -supports ne sont donc pas nécessairement coniques.

- Soit $R = \mathbb{Z}[\lambda]$ le sous-anneau de \mathbb{C} engendré par $\lambda \in \mathbb{C}$ et soit $X = \text{spec} R[x, x^{-1}] = \mathbb{A}_R^1 - \{0\} \subset \mathbb{A}_R^1 = \text{spec} R[x]$. Considérons le $D_{X/R}$ -module à gauche de type fini M correspondant à la R -connexion intégrable

$$\nabla = d + \lambda dx/x$$

sur \mathcal{O}_X . On déduit de l'identité $(x\partial)^p = x^p \partial^p + x\partial$ dans $A_1(\mathbb{Z}/p\mathbb{Z})$ [27, lemma 1] que pour tout \mathfrak{m} idéal maximal de R tel que $k(\mathfrak{m})$ soit de caractéristique positive p , le p -support de $k(\mathfrak{m}) \otimes_R M \subset T^* X'_{\mathfrak{m}} \subset T^* \mathbb{A}_{k(\mathfrak{m})}^1 = T^* \mathbb{A}_{k(\mathfrak{m})}^1$ est décrit par l'équation $xy = \lambda^p - \lambda \pmod{p}$, où y est la section globale de $\mathcal{O}_{T^* \mathbb{A}_{k(\mathfrak{m})}^1}$ correspondant à dx . Si λ n'est pas rationnel, on obtient ainsi des exemples de dépendance non-triviale en p des p -supports.

- Signalons enfin que d’après [31], les connexions de Gauss-Manin ont des p -courbures nilpotentes. Leurs p -supports sont donc réduits à la section nulle du cotangent, tout comme leur variété caractéristique.

Avant de commenter le contenu des différentes sections, attirons l’attention sur l’article [33] (voir aussi [2]), qui est consacré à des conjectures basées sur les p -supports. On s’attend notamment à ce que leur considération clarifie la théorie des \mathcal{D} -modules holonomes irréguliers.

Organisation du texte. Dans la section 1, nous fixons les notations et rappelons les faits nécessaires sur l’algèbre des opérateurs différentiels et ses modules, la géométrie symplectique du cotangent et le calcul différentiel en caractéristique positive.

Dans la section 2 figurent l’énoncé du résultat principal ainsi qu’un plan de sa démonstration (auquel on renvoie aussi pour une description du contenu de la thèse).

La section 3 traite de l’équidimension des p -supports. On y applique des techniques développées pour la variété caractéristique [20] (exposées dans [10, A :IV]), cf. aussi [30, 2.4].

La section 4 contient la réduction de la démonstration du théorème principal au cas de l’espace affine. On y utilise l’image directe des $D_{X/S}$ -modules.

Dans la section 5, on majore les rangs génériques des restrictions d’un module à ses p -supports ainsi que les degrés de ces derniers [33, conjecture 1], dans le cas de l’espace affine. Ces majorations sont essentielles aux démonstrations des théorèmes 1 et 2 énoncés ci-dessus.

La première partie de la section 6 est consacrée à la démonstration du théorème 2 sur le scindage de l’algèbre d’Azumaya des opérateurs différentiels. Dans la deuxième partie, on expose le lien entre la classe de cette algèbre dans le groupe de Brauer et la forme canonique sur le cotangent.

La section 7 contient les derniers résultats nécessaires à la démonstration du théorème principal, ainsi que la démonstration proprement dite. On s’y base sur la majoration de la section 5 pour uniformément compactifier un ouvert dense des p -supports et on étudie, suivant une suggestion de M.Kontsevich, l’action de l’opérateur de p -courbure sur l’ordre des pôles des formes différentielles.

Signalons aussi que plusieurs parties débutent par des descriptions détaillées.

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1 Preliminaries

1.1 Conventions

Schemes are assumed to be noetherian, positive characteristics to be non zero and morphisms of algebras to send $1 \mapsto 1$. Local coordinates of a smooth scheme X/S mean local étale relative coordinates in the neighborhood of a point of X . As a rule we define notions and state results for left modules, we often omit to mention that they easily adapt to right modules.

1.2 $D_{X/S}$

Let S be a scheme and let X be a smooth S -scheme of relative dimension n . Let the tangent sheaf $T_{X/S}$ be the \mathcal{O}_X -module dual to the locally free sheaf of relative differentials $\Omega_{X/S}^1$, it is endowed with a Lie bracket. The ring of PD-differential operators of X/S [6, §4], noted $D_{X/S}$, is a sheaf of noncommutative rings on X . It is the enveloping algebra of the Lie algebroid $T_{X/S}$ [8, 1.2]. Thus $D_{X/S}$ is generated by the structure sheaf \mathcal{O}_X and the tangent sheaf $T_{X/S}$, subject to relations $f.\partial = f\partial$, $\partial.f - f.\partial = \partial(f)$ and $\partial.\partial' - \partial'.\partial = [\partial, \partial']$ for f and ∂, ∂' local sections of \mathcal{O}_X and $T_{X/S}$ respectively. Note that it is compatible with base change. In terms of local étale relative coordinates $\{x_1, \dots, x_n\}$, for the sections $\{\partial_1, \dots, \partial_n\}$ of $T_{X/S}$, dual to $\{dx_1, \dots, dx_n\}$, one has $D_{X/S} = \bigoplus_I \mathcal{O}_X.\partial^I$, summing over non negative multi-indices. By definition of an enveloping algebra, endowing an \mathcal{O}_X -module with a compatible left $D_{X/S}$ -module structure or with an integrable connection are equivalent. Left multiplication by \mathcal{O}_X makes $D_{X/S}$ into a quasi-coherent \mathcal{O}_X -module. Moreover,

Proposition 1.2.1. *The sheaf of rings $D_{X/S}$ has a natural positive filtration $D_{X/S} = \bigcup_{m \geq 0} D_{X/S, \leq m}$, defined by $D_{X/S, \leq 0} := \mathcal{O}_X$ and $D_{X/S, \leq m+1} := T_{X/S}.D_{X/S, \leq m} + \overline{D}_{X/S, \leq m}$, whose associated graded sheaf of rings $grD_{X/S}$ is canonically isomorphic to $\mathcal{O}_{T^*(X/S)}$, the structure sheaf of the cotangent bundle of X/S .*

where the cotangent bundle of X/S is $T^*(X/S)/X := \text{Spec}_X(\text{Sym}_{\mathcal{O}_X} T_{X/S})$, also denoted $V(T_{X/S})$ [22, 1.7.8]. Therefore $D_{X/S}$ is a sheaf of coherent noetherian rings [3, 2.2.5 and 3.1.2]. One has the familiar finiteness condition for modules,

Proposition 1.2.2. *A left $D_{X/S}$ -module is coherent [21, 0.5.3] if and only if it is quasi-coherent as an \mathcal{O}_X -module and its module of sections over any open of an affine covering is a finitely generated left module over the ring of sections of $D_{X/S}$ [3, 3.1.3 (ii)].*

Furthermore on X affine the functor of global sections is an equivalence from the category of coherent left $D_{X/S}$ -modules to the category of finitely generated left modules over the global sections of $D_{X/S}$ [3, 3.1.3 (iii)]. Note also that coherence is preserved under base change.

Similar results hold for right $D_{X/S}$ -modules.

There is also a notion of good filtration on a coherent left $D_{X/S}$ -module [5, 5.2.3.]. Namely recall that by 1.2.1, $D_{X/S}$ is a positively filtered sheaf of rings, then

Definition 1.2.3. A filtration on a coherent left $D_{X/S}$ -module, that is a filtration by coherent sub- \mathcal{O}_X -modules compatible with the filtration on $D_{X/S}$ is said to be good if it is bounded below and if the associated graded module over $gr D_{X/S} \cong \mathcal{O}_{T^*(X/S)}$ is coherent.

Note that coherent left $D_{X/S}$ -modules admit good filtrations [5, 5.2.3 (iv)] and that for a good filtration Γ on a module M , the support of the graded coherent $gr D_{X/S} \cong \mathcal{O}_{T^*(X/S)}$ -module $gr^\Gamma M$ in $T^*(X/S)$ is independent of Γ , [11, V 2.2, lemma]. Hence the following is well-defined,

Definition 1.2.4. The singular support of a coherent left $D_{X/S}$ -module M is the support of the associated graded module to a good filtration, it is a well-defined closed subset $SS(M)$ of $T^*(X/S)$.

The singular support behaves well under exact sequences, indeed, here are some easy consequences of coherence,

Proposition 1.2.5. *Let*

$$0 \rightarrow M' \xrightarrow{\phi'} M \xrightarrow{\phi''} M'' \rightarrow 0$$

be a short exact sequence of coherent left $D_{X/S}$ -modules and let Γ be a good filtration on M . Then $gr^\Gamma M = 0$ if and only if $M = 0$. Moreover, for the induced filtrations $\Gamma' := \phi'^{-1}(\Gamma)$ on M' and $\Gamma'' := \phi''(\Gamma)$ on M'' , the natural short sequence

$$0 \rightarrow gr^{\Gamma'} M' \xrightarrow{gr \phi'} gr^\Gamma M \xrightarrow{gr \phi''} gr^{\Gamma''} M'' \rightarrow 0$$

is exact. In particular, Γ' and Γ'' are good and $SS(M) = SS(M') \cup SS(M'')$.

Assume that S is the spectrum of a field of characteristic zero, then $D_{X/S}$ is the usual algebra of differential operators D_X , also noted \mathcal{D} . Moreover the natural filtration is the usual filtration by the order of differential operators and the notions of coherent module, good filtration and singular support specialize to their \mathcal{D} -module counterparts [11, VI §1]. Recall that in characteristic zero the dimension of the singular support satisfies the following fundamental inequality, [11, VI 1.10(iii)],

Theorem 1.2.6. *Let M be a non zero coherent left \mathcal{D} -module. Then $\dim SS(M) \geq n = \dim X$.*

The modules for which this lower bound is reached are called holonomic, indeed,

Definition 1.2.7. A coherent left \mathcal{D} -module is said to be holonomic either if its singular support is of dimension $n = \dim X$ or if it is zero.

There's the following homological characterization of holonomicity, [11, VI 1.12],

Theorem 1.2.8. *A coherent left \mathcal{D} -module M is holonomic if and only if $\mathcal{E}xt_{\mathcal{D}}^i(M, \mathcal{D}) = 0$ for all $i \neq n = \dim X$.*

1.3 $D_{X/S}$ in positive characteristic

If S is a scheme of positive characteristic p , then so is X and let $X^{(p/S)}$, or simply X' , be the base change of X/S by the Frobenius endomorphism of S , raising the local sections to their p -th power. There is a S -morphism $F_{X/S} : X \rightarrow X^{(p/S)}$ associated to the Frobenius endomorphism of X and called the relative Frobenius of X/S [36, §1]. Moreover, as the p -th iterate of a derivation is again a derivation, one may associate to a local section ∂ of $T_{X/S}$ the local section $\partial^{[p]}$ of $T_{X/S}$ corresponding to its p -th iterate. Comparing it with the p -th power of the element corresponding to ∂ in $D_{X/S}$, one gets a p -linear map $c : \partial \mapsto \partial^p - \partial^{[p]}$ from $T_{X/S}$ to $D_{X/S}$, which actually lands in the center $Z(D_{X/S})$ of $D_{X/S}$ [8, 1.3.1]. By adjunction one deduces from c an $\mathcal{O}_{X'}$ -linear morphism $c' : T_{X'/S} \rightarrow F_{X/S*} Z(D_{X/S})$.

Proposition 1.3.1. ([8, 1.3.2])

The $\mathcal{O}_{X'}$ -linear morphism $c' : T_{X'/S} \rightarrow F_{X/S} Z(D_{X/S})$ extends to an $\mathcal{O}_{X'}$ -linear isomorphism $\mathcal{O}_{T^*(X'/S)} \rightarrow F_{X/S*} Z(D_{X/S})$.*

It turns $\mathcal{D}_{X/S} := F_{X/S*} D_{X/S}$ into a central $\mathcal{O}_{T^*(X'/S)}$ -algebra.

Note that in local étale coordinates as above one has $F_{X/S*} Z(D_{X/S}) = \bigoplus_I \mathcal{O}_{X'} \cdot \partial^{p^I}$ hence $\mathcal{D}_{X/S}$ is a locally free $\mathcal{O}_{T^*(X'/S)}$ -module of rank p^{2n} . Moreover,

Theorem 1.3.2. ([8, 2.2.3])

$\mathcal{D}_{X/S}$ is an Azumaya algebra of rank p^n on $T^(X'/S)$.*

Recall that an Azumaya algebra is a relative central simple algebra. The notion has many equivalent characterizations [24, 5.1] one of which is that an Azumaya algebra of rank r on a scheme Y is a sheaf of \mathcal{O}_Y -algebras, coherent as an \mathcal{O}_Y -module and isomorphic to a rank r matrix algebra $M_r(\mathcal{O}_Y)$ on a flat covering.

In the case of $\mathcal{D}_{X/S}$, let $\mathcal{A}_{X/S}$ be the centralizer of \mathcal{O}_X in $\mathcal{D}_{X/S}$ and let's denote $(\mathcal{D}_{X/S})_{\mathcal{A}_{X/S}}$ the rank p^n locally free $\mathcal{A}_{X/S}$ -module $\mathcal{D}_{X/S}$, $\mathcal{A}_{X/S}$ acting by right multiplication. Then $\mathcal{A}_{X/S}$ is a faithfully flat $F_{X/S*} Z(D_{X/S})$ -algebra and the morphism $\mathcal{D}_{X/S} \otimes_{F_{X/S*} Z(D_{X/S})} \mathcal{A}_{X/S} \rightarrow \mathcal{E}nd_{\mathcal{A}_{X/S}}((\mathcal{D}_{X/S})_{\mathcal{A}_{X/S}})$ given by left multiplication by $\mathcal{D}_{X/S}$ and right multiplication by $\mathcal{A}_{X/S}$ is an isomorphism [8, 2.2.2] thus realizing $\mathcal{D}_{X/S}$ as an Azumaya algebra of rank p^n on $T^*(X'/S)$.

1.4 Symplectic geometry of the cotangent bundle

Let S be a scheme and let Y be a smooth S -scheme of relative dimension n . Recall that the cotangent bundle of Y/S is the Y -scheme $T^*(Y/S) \xrightarrow{p_Y} Y :=$

$V((\Omega_{Y/S}^1)^*) = \text{Spec}_Y(\text{Sym}_{\mathcal{O}_Y}(\Omega_{Y/S}^1)^*)$ and hence that the sheaf of germs of Y -sections of $T^*(Y/S)/Y$ is canonically identified with $\Omega_{Y/S}^1$ [22, 1.7.9]. Moreover $T^*(Y/S)$ is a smooth Y -scheme of relative dimension n [23, 17.3.8], smooth of relative dimension $2n$ as an S -scheme. For $f : X \rightarrow Y$ a S -morphism of smooth S -schemes, the pullback of differentials $\Omega_{X/S}^1 \xleftarrow{f^*} f^*\Omega_{Y/S}^1$ [23, 16.4.3.6] gives rise to the X -morphism $T^*(X/S) \xleftarrow{f_d} X \times_Y T^*(Y/S)$ called the cotangent map. It is part of the cotangent diagram of f ,

$$\begin{array}{ccc} T^*(X/S) & \xleftarrow{f_d} & X \times_Y T^*(Y/S) \\ & & \downarrow f_\pi \\ & & T^*(Y/S) \end{array}$$

where f_π is the canonical projection. Let $U \subset Y$ be an open subset, one sees right-away on the definitions that if s_α is the section of $T^*(Y/S)/U$ corresponding to $\alpha \in \Gamma(U, \Omega_{Y/S}^1)$ then $f_d \circ X \times_Y s_\alpha$ corresponds to $(f^*)^{ad}\alpha \in \Gamma(f^{-1}U, \Omega_{X/S}^1)$.

Note the

Lemma 1.4.1. *If $f : X \rightarrow Y$ is an immersion (resp. a closed immersion) then f_d is smooth and surjective and f_π is an immersion (resp. a closed immersion). Moreover f_d admits a section locally on X .*

Proof: The morphism f_d is smooth and surjective by [23, 17.2.5], [22, 1.7.11(iii)], [23, 17.3.8] and stability under base change of surjective smooth morphisms, f_π is an immersion (resp. a closed immersion) by [21, 4.3.1(i)] and local sections of the locally split "conormal" short exact sequence of [23, 17.2.5] induce [22, 1.7.11(i)] local sections of f_d .

The cotangent bundle of Y/S carries a canonical global S -relative 1-form $\theta_{Y/S}$ corresponding to the section $T^*(Y/S) \xrightarrow{\Delta_{T^*(Y/S)/Y}} T^*(Y/S) \times_Y T^*(Y/S) \xrightarrow{(p_Y)_d} T^*(T^*(Y/S)/S)$ of the cotangent bundle $T^*(T^*(Y/S)/S) \xrightarrow{p_{T^*(Y/S)}} T^*(Y/S)$, where $\Delta_{T^*(Y/S)/Y}$ is the diagonal of $T^*(Y/S) \xrightarrow{p_Y} Y$. Let $\{y_1, \dots, y_n\}$ be local étale coordinates on Y , then in terms of the associated local étale coordinates $\{y_1, \dots, y_n; \xi_1, \dots, \xi_n\}$ on $T^*(Y/S)$, where $\{\xi_1, \dots, \xi_n\}$ are dual to $\{dy_1, \dots, dy_n\}$, $\theta_{Y/S} = \sum_{i=1}^n \xi_i dy_i$. Note that the canonical form is compatible with base change and with the cotangent diagram, the latter in the sense that $f_\pi^* \theta_{Y/S} = f_d^* \theta_{X/S}$.

Suppose that S is the spectrum of a field k and let's drop the reference to the base S from the notations. So Y is a smooth k -scheme of pure dimension n . The nondegenerate global exact 2-form $\omega_Y := d\theta_Y$ on T^*Y is called the symplectic form.

Definition 1.4.2. A subscheme $X \xrightarrow{i} T^*Y$ is said to be a lagrangian subscheme of (T^*Y, ω_Y) if it contains a dense open $U \subset X$ on which the symplectic form

vanishes, $(i^*\omega_Y)|_U = 0$ and if at each of its points x it is of dimension $n = \dim_x Y$.

We'll use the

Lemma 1.4.3. *Let $f : X \rightarrow Y$ be an immersion of smooth k -schemes and let $Z_Y \xrightarrow{i} T^*Y$ and $Z_X \xrightarrow{j} T^*X$ be reduced subschemes. Suppose that $f_\pi^{-1}Z_Y = f_d^{-1}Z_X$ and that $f_\pi^{-1}Z_Y \xrightarrow{f_\pi} Z_Y$ is surjective. Then ω_Y vanishes on a dense open subset of Z_Y if and only if ω_X vanishes on a dense open subset of Z_X .*

Proof: Note that by lemma 1.4.1, $f_d|_{Z_X}$ is smooth and surjective and $f_\pi|_{Z_Y}$ is an immersion. Since by hypothesis $f_\pi|_{Z_Y}$ is surjective, it is a nilimmersion [21, 4.5.16] hence, Z_Y being reduced, an isomorphism. Moreover by [23, 17.2.3(ii)], [12, §7 n°2 prop.4] and flatness of smooth morphisms, the pullback of forms $f_d^* : \Omega_{Z_X, z}^2 \rightarrow \Omega_{f_d^{-1}Z_X, \tilde{z}}^2$ is injective for all $z = f_d(\tilde{z})$. Since $f_\pi^*\omega_Y = f_d^*\omega_X$ as $f_\pi^*\theta_Y = f_d^*\theta_X$ and $(f_d|_{Z_X}) \circ (f_\pi|_{Z_Y})^{-1}$ and $(f_\pi|_{Z_Y}) \circ (f_d|_{Z_X})^{-1}$ preserve open dense subsets, the lemma follows.

1.5 Differential calculus in positive characteristic

Let Y be a smooth equidimensional scheme over a field k of positive characteristic p , let $Y \xrightarrow{F/k} Y'$ be the relative Frobenius of Y/k and let $W : Y' \rightarrow Y$ be the canonical projection. The differential $d = d_{Y/k} : \mathcal{O}_Y \rightarrow \Omega_{Y/k}^1$ is F/k -linear hence the differential of the complex $F/k_* \Omega_{Y/k}^\bullet$ is $\mathcal{O}_{Y'}$ -linear, where $\Omega_{Y/k}^\bullet = (\Omega_{Y/k}^\bullet, d_{Y/k} = d)$ is the de Rham complex of Y/k . Let $Z^i(F/k_* \Omega_{Y/k}^\bullet) := \ker(F/k_* d : F/k_* \Omega_{Y/k}^i \rightarrow F/k_* \Omega_{Y/k}^{i+1})$, $B^i(F/k_* \Omega_{Y/k}^\bullet) := \text{im}(F/k_* d : F/k_* \Omega_{Y/k}^{i-1} \rightarrow F/k_* \Omega_{Y/k}^i)$ and $\mathcal{H}^i(F/k_* \Omega_{Y/k}^\bullet) := Z^i(F/k_* \Omega_{Y/k}^\bullet) / B^i(F/k_* \Omega_{Y/k}^\bullet)$, the exterior product of differential forms endows $\bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(F/k_* \Omega_{Y/k}^\bullet)$ and $\bigoplus_{i \in \mathbb{Z}} \Omega_{Y'/k}^i$ with structures of graded $\mathcal{O}_{Y'}$ -algebras. They are canonically isomorphic [31, 7.2],

Theorem 1.5.1. *There is a unique morphism of graded $\mathcal{O}_{Y'}$ -algebras*

$$C_Y^{-1} : \bigoplus_{i \in \mathbb{Z}} \Omega_{Y'/k}^i \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(F/k_* \Omega_{Y/k}^\bullet)$$

such that for all local sections y of \mathcal{O}_Y , $C_Y^{-1}(d(W^*y)) =$ the class of $y^{p-1}dy \in \mathcal{H}^1(F/k_* \Omega_{Y/k}^\bullet)$. It is an isomorphism, compatible with étale localization on Y .

The composed morphism

$$\bigoplus_{i \in \mathbb{Z}} Z^i(F/k_* \Omega_{Y/k}^\bullet) \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(F/k_* \Omega_{Y/k}^\bullet) \xrightarrow{\text{inverse of } C_Y^{-1}} \bigoplus_{i \in \mathbb{Z}} \Omega_{Y'/k}^i$$

where $\bigoplus_{i \in \mathbb{Z}} Z^i(F/k_* \Omega_{Y/k}^\bullet) \rightarrow \bigoplus_{i \in \mathbb{Z}} \mathcal{H}^i(F/k_* \Omega_{Y/k}^\bullet)$ is the quotient, is denoted C_Y and called the Cartier operator.

Recall ([29, 2.1.18]) that there is an exact sequence of abelian sheaves on Y'

$$0 \rightarrow \mathcal{O}_{Y'}^* \xrightarrow{F/k_*} F/k_* \mathcal{O}_Y^* \xrightarrow{F/k_* d\log} F/k_* Z^1(\Omega_{Y/k}^\bullet) \cong Z^1(F/k_* \Omega_{Y/k}^\bullet) \xrightarrow{W^* - C_Y} \Omega_{Y'/k}^1$$

where $d\log(y) := dy/y$ for local sections y of \mathcal{O}_Y^* and W^* is induced by the morphism $F/k_* \Omega_{Y/k}^1 \xrightarrow{F/k_* ((W^*)^{ad})} F/k_* W_* \Omega_{Y'/k}^1$ coming from the pullback of forms W^* , note that since $F/k \circ W$ is the Frobenius endomorphism of Y' its underlying map is the identity and hence $F/k_* W_* \Omega_{Y'/k}^1 = (F/k \circ W)_* \Omega_{Y'/k}^1$ and $\Omega_{Y'/k}^1$ are identified as abelian sheaves on Y' . Moreover since $W^* - C_Y$ is étale locally surjective [29, 2.1.18], the above sequence induces an exact sequence of étale sheaves on Y'

$$0 \rightarrow \mathbb{G}_{m/Y'} \xrightarrow{F/k_*} F/k_* \mathbb{G}_{m/Y} \xrightarrow{F/k_* d\log} F/k_* Z^1(\Omega_{Y/k}^\bullet) \xrightarrow{W^* - C_Y} \Omega_{Y'/k}^1 \rightarrow 0$$

where for $U' \xrightarrow{f'} Y'$ étale and $U \xrightarrow{f} Y$ its base change by F/k , $F/k_* Z^1(\Omega_{Y/k}^\bullet)(U' \xrightarrow{f'} Y') := f'^* F/k_* Z^1(\Omega_{Y/k}^\bullet) \cong F/k_* Z^1(\Omega_{U/k}^\bullet)$ and $\Omega_{Y'/k}^1(U' \xrightarrow{f'} Y') := f'^* \Omega_{Y'/k}^1 \cong \Omega_{U'/k}^1$ are the étale sheaves associated to the coherent $\mathcal{O}_{Y'}$ -modules $F/k_* Z^1(\Omega_{Y/k}^\bullet)$ and $\Omega_{Y'/k}^1$ [34, II 1.6]. Let's call it the p -curvature exact sequence and $W^* - C_Y$ the p -curvature operator.

2 Statement of the result and first reductions

2.1 The p -support

Let X be a smooth scheme over a field k of positive characteristic p . Recall from 1.3, often dropping the base from the notations, that there is an $\mathcal{O}_{X'}$ -linear isomorphism $\mathcal{O}_{T^*(X')} \rightarrow F_{X/k_*} Z(D_X)$ identifying $\mathcal{O}_{T^*(X')}$ with the center $F_{X/k_*} Z(D_X)$ of $\mathcal{D}_X := F_{X/k_*} D_X$ (1.3.1). Let M be a coherent left D_X -module (1.2.2), then $\mathcal{M} := F_{X/k_*} M$ is coherent as an $\mathcal{O}_{T^*(X')}$ -module.

Definition 2.1.1. The p -support¹ of M is the support of the coherent $\mathcal{O}_{T^*(X')}$ -module $\mathcal{M} := F_{X/k_*} M$. It is a closed subset $p\text{-supp}(M)$ of $T^*(X')$, endowed with its reduced subscheme structure.

Note using [36, prop.1.a), prop.2.c)2)] and [21, 9.3.2(i)] that the p -support is compatible with étale localization on X .

Further recall that $T^*(X')$ carries a nondegenerate exact 2-form $\omega_{X'}$ which is called the symplectic form.

Remark 2.1.2. ([2, 5.2])

The symplectic form $\omega_{X'}$ or rather the corresponding Poisson bracket has a natural deformation theoretic interpretation related to the lifting of X modulo p^2 .

¹The p of p -support is the p of positive characteristic, prime characteristic and p -curvature.

2.2 The statement

Let S be a scheme of finite type over \mathbb{Z} . Recall that the closed points of S have finite residue fields and that there are lots of them, indeed every nonempty locally closed subset of S contains a closed point [23, 10.4.6 and 10.4.7].

Theorem 2.2.1. *Let S be an integral scheme dominant and of finite type over \mathbb{Z} , let X be a smooth S -scheme of relative dimension n and let M be a coherent left $D_{X/S}$ -module. Suppose that the fiber of M at the generic point of S is a holonomic left \mathcal{D} -module (1.2.7). Then there is a dense open subset U of S such that the p -support of the fiber of M at each closed point u of U is a lagrangian subscheme of $(T^*(X'_u), \omega_{X'_u})$.*

2.3 About the proof

A complete proof is given in 7.4 and rests on most of the results of the paper. Let us roughly outline the argument, which has two parts, in accordance with the definition of a lagrangian subvariety.

A first part bears on the dimension and equidimension of the p -support, notions which are of cohomological nature (vanishing of some double $\mathcal{E}xt$'s), and occupies section 3.

A second part handles the vanishing of the symplectic form on the regular locus of the p -support.

Its starting point is twofold, namely, there is a natural map from 1-forms to the Brauer group which sends the canonical form to the class of the Azumaya algebra of differential operators (subsection 6.2), and the latter splits on the regular locus of the p -support (theorem 6.1.4).

Thus the restriction of the canonical form to the regular locus of the p -support is in the kernel of the above map, which may be described in terms of the p -curvature operator (proposition 6.2.3). Further considering the action of the p -curvature operator on the order of poles along the boundary of a compactification (proposition 7.3.1), one shows that the restriction of the symplectic form, that is the exterior derivative of the canonical form, has logarithmic poles. The result then follows from the vanishing of globally exact forms with logarithmic poles, in characteristic zero.

Let us mention that crucial to the argument is an estimate of some degrees and ranks of modules, in the case of the affine space (theorem 5.3.2).

Note finally that in the above sketch, we should have written "for p large enough" several times.

2.4 First reductions

Here we carry out some standard reductions (2.4.1) and deal with an easy case of theorem 2.2.1 (2.4.3). It is organized into two remarks.

Remark 2.4.1. The conclusion of 2.2.1 depends on S only up to restricting to a dense open subset and so do its hypotheses. Moreover 2.2.1 is compatible with

Zariski (even étale) localization on X . Indeed lagrangianity and the p -support are compatible with localization while the hypotheses are stable by restriction to open coverings. Hence in the proof of 2.2.1 one may further assume that S is affine, regular (by lemma 2.4.2 below) and that X is regular [23, 17.5.8 (iii)], affine and integral [21, 4.5.7].

Lemma 2.4.2. *Let S be an integral scheme of finite type over \mathbb{Z} . Then the regular locus of S is non empty and open in S .*

Proof: The regular locus is open by [23, 6.12.6]. It is non empty since the generic stalk is a field and hence is regular.

Remark 2.4.3. If the fiber of M at the generic point of S is zero, theorem 2.2.1 is easy as there is a dense open subset U of S such that $M|_U = 0$. More generally if S is integral and M is a coherent left (resp. right) $D_{X/S}$ -module such that the fiber of M at the generic point of S is zero, then there is a dense open subset U of S such that $M|_U = 0$.

Indeed one may assume that X and S are affine and thus consider a left (resp. right) module over the ring of global sections of $D_{X/S}$. By the hypotheses, this module has a finite generating family $\{m_1, \dots, m_l\}$ and each m_i is annihilated by a non zero global section r_i of \mathcal{O}_S . Since \mathcal{O}_S acts through the center of $D_{X/S}$, the open subset of S determined by the product of these global sections fulfills the statement.

Thus we may assume that the fiber of M at the generic point of S is non zero.

3 Dimension of the p -supports

Theorem 2.2.1 boils down to assertions about the dimensions of some schemes and the vanishing of a certain 2-form. Let us start by the dimensions,

3.1 Statement

Theorem 3.1.1. *Let S be an integral scheme dominant and of finite type over \mathbb{Z} , let X be a smooth S -scheme of relative dimension n and let M be a coherent left $D_{X/S}$ -module. Suppose that the fiber of M at the generic point of S is a non zero holonomic left \mathcal{D} -module (1.2.7). Then there is a dense open subset U of S such that the p -support of the fiber of M at each closed point u of U is equidimensional of dimension $n = \dim X$.*

The proof is deferred till 3.3. It is based on the notion of pure coherent sheaf (3.2) and its characterization in terms of duality (3.2.3). This is relevant as purity implies equidimensionality (3.2.2) and holonomic \mathcal{D} -modules satisfy strong duality properties (1.2.8). One concludes using the Azumayaness of the algebra of differential operators (3.3.4).

In view of 2.4.1, we may and shall assume that S and X are regular, integral and affine.

3.2 Pure coherent sheaves

Recall that the (co)dimension of a coherent sheaf is the (co)dimension of its support and let us call a coherent sheaf equidimensional if its support is equidimensional. There is a strengthening of equidimensionality which has a very convenient interpretation in terms of duality theory. Indeed, let us fix an affine scheme Y ,

Definition 3.2.1. A coherent sheaf on Y is pure if all its non zero coherent subsheaves are of the same dimension.

It is easily seen to imply equidimensionality,

Proposition 3.2.2. *A coherent sheaf on Y is pure if and only if all its associated points [23, 3.1.1, 3.1.2] are of the same dimension. In particular a pure coherent sheaf on Y is equidimensional.*

Here's the interpretation in terms of duality theory,

Theorem 3.2.3. *Suppose that Y is regular and equidimensional. A coherent sheaf \mathcal{F} on Y is pure if and only if there's a non negative integer c such that*

$$\mathcal{E}xt^l(\mathcal{E}xt^l(\mathcal{F}, \mathcal{O}_Y), \mathcal{O}_Y) = 0$$

for all $l \neq c$. If \mathcal{F} is not zero then c is its codimension.

Proof: The proof is in the literature. Indeed, [10, A:IV 2.6] applies by [10, A:IV 3.4] and gives the result, using [11, V 2.2.3] to see that purities here and there coincide and to get the above index.

3.3 Equidimensionality of the p -supports

Recall the notations and hypotheses of 3.1.1 and use 2.4.1. In particular X/S is smooth of relative dimension n with S and X regular, affine and integral.

Here are two lemmas and two propositions, preliminary to the proof of theorem 3.1.1, which follows.

Lemma 3.3.1. *Let M be a left module over a ring R and let $\{M_i\}_{i \in \mathbb{Z}}$ be an exhaustive increasing filtration of M by left sub- R -modules. Suppose that there is i_0 such that $M_{i_0} = 0$ and for all $i > i_0$ the left R -modules M_i/M_{i-1} are flat, then M is flat. Suppose further that for all i the M_i/M_{i-1} are free, then M is free.*

Proof: By hypothesis $M_{i_0+1} \cong M_{i_0+1}/(M_{i_0} = 0)$ is flat. Since the M_i/M_{i-1} are flat for all $i \geq i_0 + 1$ and extensions of flat modules are flat [13, §2 n°5 prop.5], M_i is flat for all $i \geq i_0 + 1$. So M is a union of flat submodules, hence is flat by [13, §2 n°3 prop.2(ii)]. If the M_i/M_{i-1} are free, then the union for all $i \geq i_0 + 1$ of an arbitrary lift of a basis of M_i/M_{i-1} is a basis of M , thus M is free.

Lemma 3.3.2. *Let M be a coherent left $D_{X/S}$ -module. Then there is a dense open subset U of S such that for all l and for all $s \in U$, the canonical map*

$$(\mathcal{E}xt_{D_{X/S}}^l(M, D_{X/S}))_s \rightarrow \mathcal{E}xt_{D_{X_s}}^l(M_s, D_{X_s})$$

is an isomorphism, where the subscript s denotes restriction to the fiber.

Proof: Since coherent left $D_{X/S}$ -modules form an abelian category, the proof of [23, 9.4.3] goes through here. Indeed provided the above abelianity, the proof of [23, 9.4.2] carries reducing to associated graded to good filtrations and using [30, A.17] and lemma 3.3.1 to conclude. There are only finitely many l 's to consider since by [10, A:IV 4.5], both target and domain of the above morphism are zero for $l > \dim T^*(X/S) \geq \dim T^*X_s$, $T^*(X/S)$ and T^*X_s being the respective spectra of the regular rings $grD_{X/S}$ and grD_{X_s} .

Proposition 3.3.3. *Let M be a coherent left $D_{X/S}$ -module. Suppose that the fiber of M at the generic point of S is a holonomic left \mathcal{D} -module. Then there is a dense open subset U of S such that for all $l \neq n$ and all $s \in U$,*

$$\mathcal{E}xt_{D_{X_s}}^l(M_s, D_{X_s}) = 0.$$

Proof: By lemma 3.3.2 and theorem 1.2.8, the fiber of $\mathcal{E}xt_{D_{X/S}}^l(M, D_{X/S})$ at the generic point of S vanishes for all $l \neq n$. Hence by remark 2.4.3 and lemma 3.3.2, for each $l \neq n$ there is a dense open subset U_l of S such that for all $s \in U_l$, $\mathcal{E}xt_{D_{X_s}}^l(M_s, D_{X_s}) = 0$. Since by the proof of 3.3.2 there are at most finitely many l 's to consider, this proves the proposition.

Proposition 3.3.4. *Recall 2.1, let Y be a smooth equidimensional scheme over a field k of positive characteristic p and let M be a coherent left $D_{Y/k}$ -module. Then*

$$\mathcal{E}xt_{D_{Y/k}}^l(M, D_{Y/k}) = 0 \text{ if and only if } \mathcal{E}xt_{\mathcal{O}_{T^*(Y')}}^l(\mathcal{M}, \mathcal{O}_{T^*(Y')}) = 0,$$

where $\mathcal{M} := F_{Y/k}M$ and l is an integer.*

Proof: Since $F_{Y/k}$ is affine, $\mathcal{E}xt_{D_{Y/k}}^l(M, D_{Y/k}) = 0$ if and only if

$$0 = F_{Y/k*} \mathcal{E}xt_{D_{Y/k}}^l(M, D_{Y/k}) \cong \mathcal{E}xt_{F_{Y/k*}D_{Y/k}}^l(F_{Y/k*}M, F_{Y/k*}D_{Y/k}).$$

With the notations of 2.1, $\mathcal{E}xt_{F_{Y/k*}D_{Y/k}}^l(F_{Y/k*}M, F_{Y/k*}D_{Y/k})$ is $\mathcal{E}xt_{\mathcal{D}_Y}^l(\mathcal{M}, \mathcal{D}_Y)$. Let us show that

$$\mathcal{E}xt_{\mathcal{D}_Y}^l(\mathcal{M}, \mathcal{D}_Y) = 0 \text{ if and only if } \mathcal{E}xt_{\mathcal{O}_{T^*(Y')}}^l(\mathcal{M}, \mathcal{O}_{T^*(Y')}) = 0.$$

Both of them are coherent sheaves on $T^*(Y')$ (1.3.2) hence their respective vanishings may be checked on a flat covering $\mathcal{U} \xrightarrow{\pi} T^*(Y')$ of $T^*(Y')$, which

since \mathcal{D}_Y is an Azumaya algebra over $\mathcal{O}_{T^*(Y')}$ (1.3.2), may be chosen to split \mathcal{D}_Y , that is $(\mathcal{D}_Y)_U := \pi^* \mathcal{D}_Y \simeq M_r(\mathcal{O}_U)$, see 1.3. Moreover tensoring, $\mathcal{O}_U^r \otimes_{\mathcal{O}_U} -$, with the $(M_r(\mathcal{O}_U), \mathcal{O}_U)$ -bimodule \mathcal{O}_U^r induces an equivalence between the categories of coherent \mathcal{O}_U -modules and coherent left $M_r(\mathcal{O}_U)$ -modules. Note that the coherent sheaf $(\mathcal{O}_U^r)^*$ is sent to $\mathcal{O}_U^r \otimes_{\mathcal{O}_U} (\mathcal{O}_U^r)^* \cong M_r(\mathcal{O}_U)$ and let \mathcal{F} be a coherent sheaf sent to the coherent left $(\mathcal{D}_Y)_U \simeq M_r(\mathcal{O}_U)$ -module $\mathcal{M}_U := \pi^* \mathcal{M} \simeq \mathcal{O}_U^r \otimes_{\mathcal{O}_U} \mathcal{F}$. Then

$$\pi^* \mathcal{E}xt_{\mathcal{D}_Y}^l(\mathcal{M}, \mathcal{D}_Y) \simeq \mathcal{E}xt_{(\mathcal{D}_Y)_U}^l(\mathcal{M}_U, (\mathcal{D}_Y)_U)$$

$\simeq \mathcal{E}xt_{M_r(\mathcal{O}_U)}^l(\mathcal{O}_U^r \otimes_{\mathcal{O}_U} \mathcal{F}, \mathcal{O}_U^r \otimes_{\mathcal{O}_U} (\mathcal{O}_U^r)^*) \simeq_{\mathcal{O}_U\text{-mod}} \mathcal{E}xt_{\mathcal{O}_U}^l(\mathcal{F}, (\mathcal{O}_U^r)^*)$ vanishes if and only if $\mathcal{E}xt_{\mathcal{O}_U}^l(\mathcal{F}, \mathcal{O}_U)$ vanishes if and only if

$$\mathcal{E}xt_{\mathcal{O}_U}^l(\mathcal{O}_U^r \otimes_{\mathcal{O}_U} \mathcal{F}, \mathcal{O}_U) \simeq \mathcal{E}xt_{\mathcal{O}_U}^l(\mathcal{M}_U, \mathcal{O}_U) \simeq \pi^* \mathcal{E}xt_{\mathcal{O}_{T^*(Y')}}^l(\mathcal{M}, \mathcal{O}_{T^*(Y')}) \text{ vanishes.}$$

Proof of theorem 3.1.1: Note that if the fiber of M at the generic point of S is non zero then M is non zero. Therefore by generic freeness [18, theorem 14.4] applied to the associated graded to a good filtration on M and lemma 3.3.1, there is a dense open subset W of S on which M is faithfully flat, hence $M_s \neq 0$ and $(F_{X_s/k(s)})_* M_s \neq 0$ for all $s \in W$. By proposition 3.3.3, there is a dense open subset U of W such that $\mathcal{E}xt_{D_{X_s}}^l(M_s, D_{X_s}) = 0$ for all $l \neq n$ and all $s \in U$ which by proposition 3.3.4 is equivalent to $\mathcal{E}xt_{\mathcal{O}_{T^*(X'_s)}}^l((F_{X_s/k(s)})_* M_s, \mathcal{O}_{T^*(X'_s)}) = 0$ for all $l \neq n$ and all $s \in U$. In particular $\mathcal{E}xt_{\mathcal{O}_{T^*(X'_s)}}^l(\mathcal{E}xt_{\mathcal{O}_{T^*(X'_s)}}^l((F_{X_s/k(s)})_* M_s, \mathcal{O}_{T^*(X'_s)}), \mathcal{O}_{T^*(X'_s)}) = 0$ for all $l \neq n$ and all $s \in U$, implying by theorem 3.2.3 that for all $s \in U$, $(F_{X_s/k(s)})_* M_s$ is a pure non zero coherent $\mathcal{O}_{T^*(X'_s)}$ -module of dimension n , hence equidimensional of dimension n by 3.2.2.

Remark 3.3.5. The purity of the coherent $\mathcal{O}_{T^*(X'_s)}$ -module $(F_{X_s/k(s)})_* M_s$ guarantees that it has no embedded associated points.

4 Reduction to \mathbb{A}^n

It is convenient (see section 5) to further reduce the proof of theorem 2.2.1 to modules on \mathbb{A}_S^n . In order to do so we shall use the direct image of $D_{X/S}$ -modules.

4.1 Direct image of $D_{X/S}$ -modules for a closed immersion

Let X/S be smooth of relative dimension n , then the invertible \mathcal{O}_X -module $\Omega_{X/S} := \wedge^n \Omega_{X/S}^1$ is endowed with a right $D_{X/S}$ -module structure, defined via the Lie derivative [30, 1.4(a)] (see also [4, 1.2.1]). Moreover for M a left $D_{X/S}$ -module and for N, N' right $D_{X/S}$ -modules, $N \otimes_{\mathcal{O}_X} M$ (resp. $\mathcal{H}om_{\mathcal{O}_X}(N, N')$) is naturally a right (resp. left) $D_{X/S}$ -module, [11, VI 3.4]. In particular, denote

by M_r the right $D_{X/S}$ -module $\Omega_{X/S} \otimes_{\mathcal{O}_X} M$ and by N_l the left $D_{X/S}$ -module $\mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S}, N)$. In local étale relative coordinates $\{x_1, \dots, x_n\}$, trivializing $\Omega_{X/S}$, exchanging left and right, that is going from M to M_r and from N to N_l , is expressed by making a differential operator $P = \sum_I P_I \partial^I \in \bigoplus_I \mathcal{O}_X \cdot \partial^I \simeq D_{X/S}$ act through its adjoint $P^t := \sum_I (-1)^{|I|} \partial^I P_I$, where $|I|$ is the length $I_1 + \dots + I_n$ of the multi-index I , [4, 1.2.7].

Let Y/S be a smooth morphism of relative dimension m and let $X \xrightarrow{f} Y$ be a S -morphism. Then for a left $D_{Y/S}$ -module M , $f^*M := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}M$ is naturally endowed with a left $D_{X/S}$ -module structure, via the morphism $T_{X/S} \rightarrow f^*T_{Y/S}$ dual to the pullback of differentials $\Omega_{X/S}^1 \xleftarrow{f^*} f^*\Omega_{Y/S}^1$. In particular the pullback $f^*D_{Y/S}$ of the $(D_{Y/S}, D_{Y/S})$ -bimodule $D_{Y/S}$ is naturally a $(D_{X/S}, f^{-1}D_{Y/S})$ -bimodule, noted $D_{X \rightarrow Y}$.

Further suppose (see 2.4.1) that X, Y and S are affine. Then to a left $D_{X/S}$ -module M is associated the left $D_{Y/S}$ -module $f_0(M) := (f_*(M_r \otimes_{D_X} D_{X \rightarrow Y}))_l$. Note that f_0 is compatible with base change.

If $X \xrightarrow{f} Y$ is a closed immersion, then $f_0(M)$ is called the direct image of M by f and has the following description [11, VI §7]. In local coordinates $\{y_1, \dots, y_n, y_{n+1}, \dots, y_m\}$ of Y around a point of X in which X is described by $\{y_{n+1} = \dots = y_m = 0\}$, $f_0(M)$ is $(M_r[\partial_{n+1}, \dots, \partial_m])_l$ where we omit f_* and where $D_{Y/S} \simeq \bigoplus_I \mathcal{O}_Y \cdot \partial^I$ acts on $M_r[\partial_{n+1}, \dots, \partial_m] := M_r \otimes_{\mathcal{O}_S} \mathcal{O}_S[\partial_{n+1}, \dots, \partial_m]$ on the right using the commutation rules of $D_{Y/S}$ and the restriction $\mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ and one goes from left to right and vice-versa via the adjoint.

Hence f_0 preserves coherence and if S is of positive characteristic the above description shows that the p -support of $f_0(M)$ is $f'_\pi(f_d'^{-1}(p\text{-supp}(M)))$ where M is a coherent left $D_{X/S}$ -module, $X' \xrightarrow{f'} Y'$ is the closed immersion induced by f on the relative Frobeniuses of X and Y and we used notations of 1.4, in particular f'_π is a closed immersion by 1.4.1.

4.2 Reduction

Recall 2.4.1, in particular X/S is smooth of relative dimension n and one may assume that X and S are affine. Hence for some m , there is a closed immersion $X \xrightarrow{f} \mathbb{A}_S^m$ over S . Let M be a left $D_{X/S}$ -module as in the statement of 2.2.1, then $f_0(M)$ is a coherent left $D_{\mathbb{A}_S^m/S}$ -module (4.1) and by compatibility of direct image with base change (4.1) and [11, VI 7.8(iii)], it is holonomic at the generic fiber of S . Hence in view of the compatibility of f_0 with base change and the description of $p\text{-supp}((f_s)_0(M_s))$ for a closed point s of S given in 4.1, theorem 3.1.1 and lemma 1.4.3 further reduce the proof of theorem 2.2.1 to the case $X/S = \mathbb{A}_S^n$.

5 A bound on degrees and ranks

Thanks to 4.2, in the proof of theorem 2.2.1, one may restrict one's attention to modules M over $D_{\mathbb{A}_S^n/S}$.

In addition to the natural filtration (1.2.1), $D_{\mathbb{A}_S^n/S}$ has a filtration whose associated graded pieces are finite over S , the Bernstein filtration. We use it to refine part of the comparison (3.1.1) between fibers of M at closed points and at the generic point of S .

More precisely, we get an estimate (theorem 5.3.2), bounding the degrees of the p -supports (for a suitable projective embedding) as well as the generic ranks, of the fibers at "almost all" closed points of a module M as above. These bounds are crucial in the proofs of theorems 6.1.3 and 2.2.1.

5.1 Bernstein filtration

Let $S = \text{spec}(R)$ (2.4.1) and $\mathbb{A}_S^n = \text{spec}(R[x_1, \dots, x_n])$, then the ring of global sections of $D_{\mathbb{A}_S^n/S}$ is $R[x_1, \dots, x_n]\langle \partial_1, \dots, \partial_n \rangle / \langle [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{i,j} \rangle$, the n -th Weyl algebra with coefficients in R , $A_n(R)$. The filtration on $A_n(R) = \bigoplus_{\alpha, \beta} R x^\alpha \partial^\beta$, α, β multi-indices, by the total order in x and ∂ , $\mathcal{B}_l A_n(R) := \bigoplus_{|\alpha|+|\beta| \leq l} R x^\alpha \partial^\beta$ is called the *Bernstein filtration*. Note that the associated graded ring $gr^{\mathcal{B}} A_n(R)$ is the R -algebra of polynomials in the classes $x_1, \dots, x_n, y_1, \dots, y_n$ of $x_1, \dots, x_n, \partial_1, \dots, \partial_n$, respectively, graded by the order of polynomials. In particular the $\mathcal{B}_l A_n(R)/\mathcal{B}_{l-1} A_n(R)$ are finite free modules over R .

A good filtration Γ on a left $A_n(R)$ -module M is an increasing exhaustive filtration on M , compatible with \mathcal{B} , which is bounded below and such that the associated graded module $gr^\Gamma M$ is finite over the algebra of polynomials $gr^{\mathcal{B}} A_n(R)$. In particular the $\Gamma_l M / \Gamma_{l-1} M$ and hence the $\Gamma_l M$ are finite R -modules. Note that good filtrations exist on finitely generated left $A_n(R)$ -modules, [9, Ch.1 2.7].

Suppose that R is a field K . Let M be a finitely generated left $A_n(K)$ -module and let Γ be a good filtration on M . Then for l large enough, the function $l \mapsto \dim_K \Gamma_l M$ coincides with a polynomial $\mathcal{H}_{M, \Gamma} \in \mathbb{Q}[t]$, [9, Ch.1 3.3]. Moreover, let d (resp. a_d) be the degree (resp. the leading coefficient) of $\mathcal{H}_{M, \Gamma}$, then $d!a_d$ is a non negative integer and $d(M) := d$ and $e(M) := d!a_d$ are independent of Γ and called the dimension and multiplicity of M , respectively [9, p.8].

Lemma 5.1.1. *Suppose that R is a domain and let M be a finitely generated left $A_n(R)$ -module. Then there is a dense open subset U of $S := \text{spec}(R)$ such that the functions $s \mapsto d(M_s)$ and $s \mapsto e(M_s)$ are constant on U .*

Proof: Let Γ be a good filtration on M . Then by generic freeness [18, theorem 14.4], there is a dense open subset U of S such that for all l , $(\Gamma_l M / \Gamma_{l-1} M)|_U$ is free over U . In particular the $(\Gamma_l M / \Gamma_{l-1} M)|_U$ are flat over U , hence for all l and all $s \in U$, $(\Gamma_l M / \Gamma_{l-1} M)_s \cong (\Gamma_l M)_s / (\Gamma_{l-1} M)_s$ and $(\Gamma)_s$ is a good filtration on M_s . The lemma follows since for all $s \in U$ and all l , $\dim_{k(s)} (\Gamma_l M)_s =$

$\sum_{i=-\infty}^{i=l} \dim_{k(s)}(\Gamma_i M)_s / (\Gamma_{i-1} M)_s$ and

$$\dim_{k(s)}(\Gamma_l M)_s / (\Gamma_{l-1} M)_s = \dim_{k(s)}(\Gamma_l M / \Gamma_{l-1} M)_s$$

is the rank of the free module $\Gamma_l M / \Gamma_{l-1} M|_U$ over U .

5.2 Induced filtration over the center

Let K be a field of positive characteristic p . Then the center $ZA_n(K)$ of $A_n(K) = K[x_1, \dots, x_n] \langle \partial_1, \dots, \partial_n \rangle / \langle [\partial_i, \partial_j] = 0, [\partial_i, x_j] = \delta_{i,j} \rangle$ is the algebra of polynomials $K[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]$. It is graded by the degree of polynomials, where $\deg(x_i^p) = \deg(\partial_j^p) = 1$ and the associated increasing filtration is denoted \mathcal{C} . The Rees ring $R_n(\mathcal{C})$ of the filtered ring $(ZA_n(K), \mathcal{C})$ is the naturally graded ring $\bigoplus_{i=0}^{i=\infty} \mathcal{C}_i ZA_n(K)$. Note that the graded algebra morphism $K[t_0, x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p] \rightarrow R_n(\mathcal{C}) := \bigoplus_{i=0}^{i=\infty} \mathcal{C}_i K[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]$ sending $t_0 \mapsto 1 \in \mathcal{C}_1 K[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]$ and x_i^p (resp. ∂_j^p) $\mapsto x_i^p$ (resp. ∂_j^p) $\in \mathcal{C}_1 K[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]$ is an isomorphism. Keeping the same notation for t_0 and its image under this isomorphism, the natural map $R_n(\mathcal{C})/t_0 R_n(\mathcal{C}) \rightarrow gr^{\mathcal{C}} ZA_n(K)$ is an isomorphism of graded algebras. Note also that summing components induces an isomorphism $R_n(\mathcal{C})_{(t_0)} \xrightarrow{\sim} ZA_n(K)$ where $R_n(\mathcal{C})_{(t_0)}$ is the subring of degree 0 elements of the graded ring $R_n(\mathcal{C})_{t_0}$.

An increasing \mathcal{C} -compatible filtration G on a $ZA_n(K)$ -module M is said to be good if the associated Rees module $\mathcal{R}(M, G) := \bigoplus_{i=-\infty}^{i=\infty} G_i M$ over the Rees ring $\bigoplus_{i=0}^{i=\infty} \mathcal{C}_i ZA_n(K)$ is finitely generated. This implies in particular that G is bounded below. Moreover one easily sees that a filtration G on M is good if and only if G is bounded below and the associated graded module $gr^G M$ is finitely generated over $gr^{\mathcal{C}} ZA_n(K)$, [10, A:III 1.29].

Let Γ be a filtration on the left $A_n(K)$ -module M , then $p\Gamma$, $(p\Gamma)_l M := \Gamma_{pl} M$ endows M with the structure of a filtered module over the center $(ZA_n(K), \mathcal{C})$.

Lemma 5.2.1. *Let Γ be a good filtration on the left $A_n(K)$ -module M , then $p\Gamma$ is a good filtration on M seen as a $(ZA_n(K), \mathcal{C})$ -module.*

Proof: Since Γ is bounded below then so is $p\Gamma$. Let's show that the $gr^{\mathcal{C}} ZA_n(K)$ -module $gr^{p\Gamma} M$ is finitely generated. For this let F be the filtration on $ZA_n(K)$ induced by \mathcal{B} , in particular $\deg_F(x_i^p) = \deg_F(\partial_j^p) = p$ and let $F(\Gamma)$ be the F -compatible filtration on M defined by $F(\Gamma)_l M := \Gamma_{pm} M$, where pm is the greatest integer multiple of p such that $pm \leq l$. Note that class of x_i^p (resp. ∂_j^p) \mapsto class of x_i^p (resp. ∂_j^p) induces an isomorphism of K -algebras $gr^{\mathcal{C}} ZA_n(K) \rightarrow gr^F ZA_n(K)$ with which the K -module isomorphism $gr^{p\Gamma} M \rightarrow gr^{F(\Gamma)} M$ defined by $(p\Gamma)_l M / (p\Gamma)_{l-1} M = \Gamma_{pl} M / \Gamma_{p(l-1)} M = F(\Gamma)_{pl} M / F(\Gamma)_{p(l-1)} M$ is compatible. Hence $gr^{p\Gamma} M$ is finitely generated over $gr^{\mathcal{C}} ZA_n(K)$ if and only if $gr^{F(\Gamma)} M$ is finitely generated over $gr^F ZA_n(K)$. Consider the finite exhaustive filtration of $gr^{F(\Gamma)} M$ by graded sub- $gr^F ZA_n(K)$ -modules, $0 = (gr^{F(\Gamma)} M)_0 \subset (gr^{F(\Gamma)} M)_1 \subset \dots \subset (gr^{F(\Gamma)} M)_p = gr^{F(\Gamma)} M$

such that $(gr^{F(\Gamma)}M)_i \cap F(\Gamma)_l M / F(\Gamma)_{l-1} M$ is the image of the natural map $\Gamma_{p(m-1)+i} M / (F(\Gamma)_{l-1} M \cap \Gamma_{p(m-1)+i} M) \rightarrow F(\Gamma)_l M / F(\Gamma)_{l-1} M$ with pm , as above, the greatest integer multiple of p such that $pm \leq l$. Denote by $gr(gr^{F(\Gamma)}M)$ the graded $gr^F ZA_n(K)$ -module $\bigoplus_{i=1}^{i=p} (gr^{F(\Gamma)}M)_i / (gr^{F(\Gamma)}M)_{i-1}$. Note also that $gr^\Gamma M$ seen as a module over $gr^F ZA_n(K) \hookrightarrow gr^B A_n(K)$ decomposes as a direct sum of graded sub- $gr^F ZA_n(K)$ -modules $\bigoplus_{i=1}^{i=p} (gr^\Gamma M)_i$ where $(gr^\Gamma M)_i := \bigoplus_{l \in \mathbb{Z}} gr_{pl+i}^\Gamma M$ and let $F_* gr^\Gamma M$ be the graded $gr^F ZA_n(K)$ -module $\bigoplus_{i=1}^{i=p} (gr^\Gamma M)_i[i-p]$. Then for all i , the morphism of graded $gr^F ZA_n(K)$ -modules $(gr^{F(\Gamma)}M)_i / (gr^{F(\Gamma)}M)_{i-1} \rightarrow (gr^\Gamma M)_i[i-p]$ induced by $\Gamma_{p(m-1)+i} M \rightarrow (gr^\Gamma M)_i[i-p]$ in degree $pm = (gr^\Gamma M)_i$ in degree $p(m-1)+i = gr_{p(m-1)+i}^\Gamma M$ is an isomorphism. These induce an isomorphism of graded $gr^F ZA_n(K)$ -modules

$$gr(gr^{F(\Gamma)}M) \simeq F_* gr^\Gamma M.$$

Since $gr^\Gamma M$ is finitely generated over $gr^B A_n(K)$ by hypothesis and $gr^B A_n(K) \cong K[x_1, \dots, x_n, y_1, \dots, y_n]$ is finite over $gr^F ZA_n(K) \cong K[x_1^p, \dots, x_n^p, y_1^p, \dots, y_n^p]$, $F_* gr^\Gamma M$ is finitely generated over $gr^F ZA_n(K)$. Then $gr^{F(\Gamma)}M$ has an exhaustive finite filtration whose subquotients are finitely generated over $gr^F ZA_n(K)$, hence it is finitely generated, giving the lemma.

Let M be a left $A_n(K)$ -module and let Γ be a good filtration on M . Since $p\Gamma$ is a good filtration on the $(ZA_n(K), \mathcal{C})$ -module M (5.2.1) the Rees module of $(M, p\Gamma)$, which is a finitely generated graded module over the Rees ring $R_n(\mathcal{C}) \simeq K[t_0, x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]$, has a Hilbert polynomial $\mathcal{H}_{\mathcal{R}(M, p\Gamma)}$. The latter can be computed in terms of the Hilbert polynomial of (M, Γ) and in particular one gets the degree and leading coefficient of $\mathcal{H}_{\mathcal{R}(M, p\Gamma)}$ in terms of the dimension and multiplicity of M . Indeed,

Proposition 5.2.2. *Let M be a left $A_n(K)$ -module and let Γ be a good filtration on M . Then the Hilbert polynomial $\mathcal{H}_{\mathcal{R}(M, p\Gamma)}(t)$ of the Rees module of $(M, p\Gamma)$ is $\mathcal{H}_{M, \Gamma}(pt)$. In particular, the degree of $\mathcal{H}_{\mathcal{R}(M, p\Gamma)}$ is $d(M)$ and its leading coefficient times $d(M)!$ is $e(M)p^{d(M)}$.*

Proof: For l large enough, the Hilbert polynomial $\mathcal{H}_{M, \Gamma}(l)$ coincides with $l \mapsto \dim_K \Gamma_l M$ while $\mathcal{H}_{\mathcal{R}(M, p\Gamma)}(l)$ does so with $l \mapsto \dim_K (p\Gamma)_l M = \dim_K \Gamma_{pl} M$. The proposition follows.

5.3 The bound

There is a geometric picture of the Rees construction in which the affine scheme $\text{spec}(ZA_n(K)) \cong \text{spec}(R_n(\mathcal{C})_{(t_0)})$ is identified with the affine open $D_+(t_0)$ associated to t_0 in the homogeneous prime spectrum $\text{Proj}(R_n(\mathcal{C}))$ of $R_n(\mathcal{C})$ [22, 2.4.1] and in which its complement, the reduced closed subscheme $V_+(t_0)$ is identified with the closed subscheme $\text{Proj}(gr^C ZA_n(K)) \cong \text{Proj}(R_n(\mathcal{C})/t_0 R_n(\mathcal{C}))$ of $\text{Proj}(R_n(\mathcal{C}))$, [22, 2.9.2 (i)]. Making these identifications, let G be a good filtration on a finitely generated $(ZA_n(K), \mathcal{C})$ -module M then the coherent sheaf

$\widetilde{\mathcal{R}(M, G)}$ on $\text{Proj}(R_n(\mathcal{C}))$ extends \widetilde{M} and its restriction to the complement $\text{Proj}(\text{gr}^{\mathcal{C}} Z A_n(K))$ of $\text{spec}(Z A_n(K))$ is isomorphic to $\widetilde{\text{gr}^G M}$. Moreover it's easy to see that the support of $\widetilde{\mathcal{R}(M, G)}$ is the closure of $\text{supp}(\widetilde{M})$ in $\text{Proj}(R_n(\mathcal{C}))$. Note that here

$$\begin{aligned} \text{Proj}(R_n(\mathcal{C})) &\simeq \text{Proj}(K[t_0, x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]) \simeq \mathbb{P}_K^{2n}, \\ \text{spec}(Z A_n(K)) &\simeq \text{spec}(K[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]) \simeq \mathbb{A}_K^{2n} \text{ and} \\ \text{Proj}(\text{gr}^{\mathcal{C}} Z A_n(K)) &\simeq \text{Proj}(K[x_1^p, \dots, x_n^p, \partial_1^p, \dots, \partial_n^p]) \simeq \mathbb{P}_K^{2n-1}. \end{aligned}$$

The leading coefficient of the Hilbert polynomial of $\widetilde{\mathcal{R}(M, G)}$ is related to the top-dimensional irreducible components of its support through the following,

Proposition 5.3.1. *Let $Y \xrightarrow{i} \mathbb{P}_K^m$ be a closed subscheme and let \mathcal{F} be a coherent sheaf of dimension d on Y . Set $\mu(\mathcal{F}) := d!a_d$ where a_d is the leading coefficient of the Hilbert polynomial of \mathcal{F} with respect to i . Then*

$$\sum_z \text{rk}_z(\mathcal{F}) \deg(\overline{\{z\}}) \leq \mu(\mathcal{F})$$

where the sum is over the generic points of the d -dimensional irreducible components of $\text{supp}(\mathcal{F})$, $\text{rk}_z(\mathcal{F}) := \dim_{k(z)}(\mathcal{F}_z \otimes k(z))$ and $\deg(\overline{\{z\}})$ is the degree of $\overline{\{z\}}^{\text{red}}$ with respect to i .

Proof: By [32, lemma B.4] and [23, 5.3.1], $\mu(\mathcal{F}) = \sum_z \text{length}_{\mathcal{O}_{Y,z}}(\mathcal{F}_z) \mu(\mathcal{O}_{\overline{\{z\}}^{\text{red}}})$ summing over the generic points of the d -dimensional irreducible components of $\text{supp}(\mathcal{F})$. Let z be as above, then by additivity of the length under short exact sequences $\text{length}_{\mathcal{O}_{Y,z}}(\mathcal{F}_z) \geq \text{length}_{k(z)}(\mathcal{F}_z \otimes k(z)) = \dim_{k(z)}(\mathcal{F}_z \otimes k(z)) =: \text{rk}_z(\mathcal{F})$. This gives the proposition since $\deg(\overline{\{z\}}) := \mu(\mathcal{O}_{\overline{\{z\}}^{\text{red}}})$.

Theorem 5.3.2. *Let S be an integral scheme dominant and of finite type over \mathbb{Z} and let M be a coherent left $D_{\mathbb{A}^n_S}/S$ -module. Suppose that the fiber of M at the generic point of S is a non zero holonomic left \mathcal{D} -module. Then there is a dense open subset U of S such that for each closed point $u \in U$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$*

$$\deg(\overline{\{z\}}) \leq e(M) \text{ and } \text{rk}_z(M_u) \leq e(M)p^n$$

where $e(M)$ is the multiplicity for the Bernstein filtration of the fiber of M at the generic point of S , $\deg(\overline{\{z\}})$ is the degree of the reduced closure of the image of z in $\mathbb{P}_{k(u)}^{2n}$ by the open immersion of the Rees construction and $\text{rk}_z(M_u) := \dim_{k(z)}((F_{\mathbb{A}^n/k(u)_*} M_u)_z \otimes k(z))$.

Proof: The proof reduces to the case S is integral and affine = $\text{spec}(R)$. Hence it is equivalent (1.2.2) to consider a finitely generated left $A_n(R)$ -module M . By 5.1.1 there is a dense open subset U_e of S such that for each closed

point $u \in U_e$, $d(M_u) = n$ and $e(M_u) = e(M)$. Then for such a u set $p := \text{char}(k(u))$ and let Γ be a good filtration on the left $A_n(k(u))$ -module M_u . By 5.2.2 $\mathcal{R}(\widetilde{M_u, p\Gamma})$ is of dimension n and $\mu(\mathcal{R}(\widetilde{M_u, p\Gamma})) = e(M)p^n$. Hence since $\text{supp}(\mathcal{R}(\widetilde{M_u, p\Gamma})) = \overline{p\text{-supp}(M_u)}$ in which $p\text{-supp}(M_u) = \overline{p\text{-supp}(M_u)} \cap \text{spec}(ZA_n(K))$ is open, 5.3.1 gives $\sum_z rk_z(M_u) \deg(\{z\}) \leq e(M)p^n$ where the sum is over the generic points of the n -dimensional irreducible components of $p\text{-supp}(M_u)$, which are all its irreducible components if $u \in U \subset U_e$ where $U \subset U_e$ is a dense open subset provided by theorem 3.1.1. In particular for each z generic point of an irreducible component of $p\text{-supp}(M_u)$, $rk_z(M_u) \deg(\{z\}) \leq e(M)p^n$.

Moreover $F_{\mathbb{A}^n/k(u)_*} M_u$ being a left module over an Azumaya algebra of rank p^n (1.3.2), $(F_{\mathbb{A}^n/k(u)_*} M_u)_z \otimes \overline{k(z)}$ is by [24, 5.1 (i)] a left module over $M_{p^n}(\overline{k(z)})$, where $\overline{k(z)}$ is an algebraic closure of $k(z)$. Hence there is a finite dimensional $\overline{k(z)}$ -vector space V such that $(F_{\mathbb{A}^n/k(u)_*} M_u)_z \otimes \overline{k(z)} \simeq \overline{k(z)}^{p^n} \otimes_{\overline{k(z)}} V$ where $\overline{k(z)}^{p^n}$ is the standard left $M_{p^n}(\overline{k(z)})$ -module. In particular

$$rk_z(M_u) := \dim_{k(z)}((F_{\mathbb{A}^n/k(u)_*} M_u)_z \otimes k(z)) = \dim_{\overline{k(z)}}((F_{\mathbb{A}^n/k(u)_*} M_u)_z \otimes \overline{k(z)})$$

is divisible by p^n , thus proving the theorem.

6 The Brauer group and differential forms

Here we prove, in a first part, that "the Azumaya algebra of differential operators splits on the regular locus of the p -support of a holonomic \mathcal{D} -module, for p large enough" (theorem 6.1.4).

In a second part, we consider a map arising from the p -curvature exact sequence (1.5), which sends 1-forms to the Brauer group. It maps the canonical form to the class of the Azumaya algebra of differential operators (prop. 6.2.4) and we describe its kernel (prop. 6.2.3).

6.1 Splittings of Azumaya algebras on the support of modules

Let Y be a scheme and let \mathcal{A} be an Azumaya algebra on Y . Since \mathcal{A} is a coherent \mathcal{O}_Y -module, it is a coherent noetherian ring and a left \mathcal{A} -module is coherent if and only if it is coherent as an \mathcal{O}_Y -module. The Azumaya algebra \mathcal{A} is said to split on Y if its class $[\mathcal{A}]$ in the Brauer group $Br(Y)$ of Y [24, §2] is trivial.

Let M be a coherent left \mathcal{A} -module and let z be the generic point of an irreducible component of the support of the coherent \mathcal{O}_Y -module M . The next proposition relates $rk_z(M)$ (5.3.1) to the order of $[\mathcal{A}|_{\overline{\{z\}}^{red}}]_{reg}$ in $Br((\overline{\{z\}}^{red})^{reg})$.

Let us first prove a lemma,

Lemma 6.1.1. *Let Y be a scheme and let \mathcal{A} be an Azumaya algebra of rank r on Y . Suppose that \mathcal{A} acts on the left on a locally free sheaf \mathcal{V} of rank v . Then r divides $v = lr$ and $l[\mathcal{A}] = 0$ in $Br(Y)$.*

Proof: By hypothesis there is a morphism of \mathcal{O}_Y -algebras $\mathcal{A} \rightarrow \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})$, $1 \mapsto$
1. It is injective by [21, 0.5.5.4] since the fiber of \mathcal{A} at each point of Y is a simple algebra [24, 5.1 (i)]. Therefore one may view \mathcal{A} as a subalgebra of $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})$ and in particular consider $\mathcal{C}_{\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})}(\mathcal{A})$ the commutant of \mathcal{A} in $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})$, which is a coherent subalgebra of $\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})$. By [1, theorem 3.3], the natural morphism of \mathcal{O}_Y -algebras $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{C}_{\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})}(\mathcal{A}) \rightarrow \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})$ is an isomorphism and $\mathcal{C}_{\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})}(\mathcal{A})$ is an Azumaya algebra on Y . Hence by the behaviour of ranks under tensor products, $\mathcal{C}_{\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})}(\mathcal{A})$ is of constant rank l , $v = lr$. By definition of the Brauer group $0 = [\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})] = [\mathcal{A}] + [\mathcal{C}_{\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})}(\mathcal{A})]$ in $Br(Y)$. The lemma follows since for each Azumaya algebra \mathcal{B} of rank n on Y , $n[\mathcal{B}] = 0$ in $Br(Y)$ [24, §2] giving $0 = l[\mathcal{A}] + l[\mathcal{C}_{\mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{V})}(\mathcal{A})] = l[\mathcal{A}]$ in $Br(Y)$.

Proposition 6.1.2. *Suppose that Y is of finite type over a field K . Let \mathcal{A} be an Azumaya algebra of rank r on Y , let M be a coherent left \mathcal{A} -module and let z be the generic point of an irreducible component of $\text{supp}(M)$. Then r divides $rk_z(M) = l_z(M)r$ and*

$$l_z(M)[\mathcal{A}|_{(\overline{\{z\}}^{red})^{reg}}] = 0$$

in $Br((\overline{\{z\}}^{red})^{reg})$.

Proof: Since the vector space $M_z \otimes k(z)$ is of dimension $rk_z(M)$ and acted upon on the left by the rank r Azumaya algebra $\mathcal{A}_z \otimes k(z)$, lemma 6.1.1 implies that $rk_z(M) = l_z(M)r$ and $l_z(M)[\mathcal{A}_z \otimes k(z)] = 0$ in $Br(k(z))$. Moreover since Y is of finite type over a field, so is $\overline{\{z\}}^{red}$ and $(\overline{\{z\}}^{red})^{reg} \hookrightarrow \overline{\{z\}}^{red}$ is a non empty open subscheme by [23, 6.12.5]. Hence $\mathcal{A}_z \otimes k(z) \cong (\mathcal{A}|_{\overline{\{z\}}^{red}})_z \otimes k(z) \cong (\mathcal{A}|_{(\overline{\{z\}}^{red})^{reg}})_z \otimes k(z)$ and the proposition follows from the canonical embedding $Br((\overline{\{z\}}^{red})^{reg}) \hookrightarrow Br(k(z))$ [34, IV 2.6].

The above proposition combined with the second estimate of theorem 5.3.2 leads to the

Theorem 6.1.3. *Let S be an integral scheme dominant and of finite type over \mathbb{Z} and let M be a coherent left $D_{\mathbb{A}_S^n/S}$ -module. Suppose that the fiber of M at the generic point of S is a non zero holonomic left \mathcal{D} -module. Then there is a dense open subset U of S such that for each closed point $u \in U$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$ the Azumaya algebra $F_{\mathbb{A}^n/k(u)*} D_{\mathbb{A}_{k(u)}^n}$ on $T^*(\mathbb{A}_{k(u)}^{n'})$ splits on $(\overline{\{z\}}^{red})^{reg}$.*

Proof: By theorem 5.3.2 and using its notations, there is a dense open subset U_b of S such that for each closed point $u \in U_b$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$, $rk_z(M_u) \leq e(M)p^n$ where $p := \text{chark}(u)$.

Moreover by proposition 6.1.2, $rk_z(M_u) = l_z(M_u)p^n \leq e(M)p^n$, thus $l_z(M_u) \leq e(M)$ and $l_z(M_u)[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}] = 0$ in $Br((\overline{\{z\}}^{red})^{reg})$. Note that by definition $l_z(M_u) \neq 0$ and hence for $u \in U$ the open dense subset of U_b defined by inverting all the primes $\leq e(M)$, $l_z(M_u)$ and p^n are coprime, that is there are integers a and b such that $1 = a.l_z(M_u) + b.p^n$. Since $F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n}$ is of rank p^n , $p^n[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}] = 0$ by [24, §2] and the theorem follows from $[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}] = 1[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}]$

$$\begin{aligned}
&= (a.l_z(M_u) + b.p^n)[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}] \\
&= a.l_z(M_u)[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}] + b.p^n.[F_{\mathbb{A}^n/k(u)} * D_{\mathbb{A}_{k(u)}^n} |_{(\overline{\{z\}}^{red})^{reg}}] \\
&= 0 \text{ in } Br((\overline{\{z\}}^{red})^{reg}).
\end{aligned}$$

As in 4.2, theorem 6.1.3 implies the apparently more general

Theorem 6.1.4. *Let S be an integral scheme dominant and of finite type over \mathbb{Z} , let X be a smooth S -scheme of relative dimension n and let M be a coherent left $D_{X/S}$ -module. Suppose that the fiber of M at the generic point of S is a holonomic left \mathcal{D} -module. Then there is a dense open subset U of S such that for each closed point $u \in U$ the Azumaya algebra $F_{X_u/k(u)} * D_{X_u}$ on $T^*(X'_u)$ splits on $(p\text{-supp}(M_u))^{reg}$.*

Proof: By [24, 2.1] applied to the Zariski site and [23, 21.11.1] the case $i = 2$ of lemma 6.1.5 below implies that on a regular (noetherian) scheme for an Azumaya algebra to be split is a Zariski local condition. Therefore by remark 2.4.1 one may further assume that S and X are regular integral and affine and in particular that there is a closed immersion $X \xrightarrow{f} \mathbb{A}_S^m$ over S . Specializing to a closed point u of positive characteristic p of S it follows from the description of $p\text{-supp}(f_0(M_u))$ in 4.1 and from [23, 17.5.8(iii)] f'_d being smooth that $p\text{-supp}(f_0(M_u))^{reg} = f'_\pi(f_d'^{-1}((p\text{-supp}(M_u))^{reg}))$. Moreover by [7, 3.7] $f_d'^*(F_{X_u/k(u)} * D_{X_u})$ splits on $f_d'^{-1}(p\text{-supp}(M_u))^{reg}$ if $F_{\mathbb{A}^m/k(u)} * D_{\mathbb{A}_{k(u)}^m}$ splits on $p\text{-supp}(f_0(M_u))^{reg}$. Since by lemma 1.4.1 f'_d Zariski locally admits a section there is a Zariski open covering of X' above which the pullback of Brauer classes $f_d'^*$ is injective and therefore to split being Zariski local on a regular noetherian scheme, $f_d'^*$ induces an injective morphism $Br(p\text{-supp}(M_u))^{reg} \rightarrow Br(f_d'^{-1}(p\text{-supp}(M_u))^{reg})$. So $F_{X_u/k(u)} * D_{X_u}$ splits on $p\text{-supp}(M_u)^{reg}$ if $F_{\mathbb{A}^m/k(u)} * D_{\mathbb{A}_{k(u)}^m}$ splits on $p\text{-supp}(f_0(M_u))^{reg}$. Note that if the fiber of M at the generic point of S is zero the theorem holds by remark 2.4.3 and that a regular noetherian scheme is the sum of its irreducible components by [21, 2.1.9(iii)], in particular an Azumaya algebra splits on a regular noetherian scheme if and only if it splits on its irreducible components. Thus since by 4.1 the fiber of $f_0(M)$ at the generic point of S is non zero if so is that of M and since $f_0(M)$ satisfies the other hypotheses of 6.1.3 by 4.2, 6.1.4 reduces to theorem 6.1.3.

Lemma 6.1.5. *Let Y be a noetherian scheme. If Y is locally factorial then the Zariski cohomology $H^i(Y, \mathcal{O}_Y^*) = 0$ for all $i \geq 2$.*

Proof: By definition of the sheaf $\mathcal{D}iv_Y$ of Cartier divisors there's an exact sequence of abelian sheaves $0 \rightarrow \mathcal{O}_Y^* \rightarrow \mathcal{K}_Y^* \rightarrow \mathcal{D}iv_Y \rightarrow 0$ on Y where \mathcal{K}_Y is the sheaf of meromorphic functions and $\mathcal{O}_Y^* \rightarrow \mathcal{K}_Y^*$ is the natural injection. If Y is locally factorial then it is the sum of its (finitely many) irreducible components, each of which is integral [21, 4.5.5]. Hence if $Y_i \xrightarrow{f_i} Y$ is the open immersion of the i -th irreducible component then $\mathcal{K}_Y^* \cong \Pi_i f_{i*} \mathcal{K}_{Y_i}^*$ where $\mathcal{K}_{Y_i}^*$ is isomorphic to the constant sheaf associated to $k(y_i)^*$ for y_i the generic point of Y_i . In particular \mathcal{K}_Y^* is flasque. Since $\mathcal{D}iv_Y$ is flasque by [23, 21.6.11], $\mathcal{K}_Y^* \rightarrow \mathcal{D}iv_Y$ is a flasque right resolution of \mathcal{O}_Y^* . This gives the result as sheaf cohomology may be computed using flasque resolutions.

6.2 The Brauer group via the p -curvature sequence

Let Y be a smooth equidimensional scheme over a perfect field K of positive characteristic p . Composing the coboundary morphisms of the étale cohomology long exact sequences of the two short exact sequences of étale sheaves on Y' , $0 \rightarrow \mathbb{G}_{m/Y'} \xrightarrow{F/K^*} F/K_* \mathbb{G}_{m/Y} \xrightarrow{F/K_* dlog} Im F/K_* dlog \rightarrow 0$ and $0 \rightarrow coker F/K_* \xrightarrow{F/K_* dlog} F/K_* Z^1(\Omega_{Y/K}^\bullet) \xrightarrow{W^* - C_Y} \Omega_{Y'/K}^1 \rightarrow 0$ deduced from the p -curvature exact sequence (1.5)

$$0 \rightarrow \mathbb{G}_{m/Y'} \xrightarrow{F/K^*} F/K_* \mathbb{G}_{m/Y} \xrightarrow{F/K_* dlog} F/K_* Z^1(\Omega_{Y/K}^\bullet) \xrightarrow{W^* - C_Y} \Omega_{Y'/K}^1 \rightarrow 0,$$

one gets a morphism $H^0(Y', \Omega_{Y'/K}^1) \rightarrow H^1(Y', coker F/K_* \cong Im F/K_* dlog) \rightarrow H^2(Y', \mathbb{G}_{m/Y'})$ which by construction factors through

$$H^0(Y', \Omega_{Y'/K}^1) \rightarrow coker H^0(W^* - C_Y) \rightarrow ker H^2(F/K^*) \hookrightarrow H^2(Y', \mathbb{G}_{m/Y'}),$$

where $H^0(Y', \Omega_{Y'/K}^1) \rightarrow coker H^0(W^* - C_Y)$ and $ker H^2(F/K^*) \hookrightarrow H^2(Y', \mathbb{G}_{m/Y'})$ are the canonical coker and ker morphisms. Since by [28, 2.1] the canonical embedding $Br(Y') \hookrightarrow H^2(Y', \mathbb{G}_{m/Y'})$ [24, 2.1] is an isomorphism on the p -torsion ($:=$ the kernel of multiplication by p) $Br(Y')_p \xrightarrow{\sim} H^2(Y', \mathbb{G}_{m/Y'})_p = ker H^2(F/K^*)$, this leads to a morphism

$$\phi_Y : H^0(Y', \Omega_{Y'/K}^1) \rightarrow coker H^0(W^* - C_Y) \xrightarrow{\overline{\phi_Y}} Br(Y')_p \subset Br(Y').$$

Remark 6.2.1. Here's another description of ϕ_Y [35, Rem. 4.3]. Let $\alpha \in H^0(Y', \Omega_{Y'/K}^1)$, then $\phi_Y(\alpha) = [s_\alpha^*(F_{Y/K_*} D_Y)] \in Br(Y')$ where $Y' \xrightarrow{s_\alpha} T^*(Y'/K)$ is the section of $T^*(Y'/K)/Y'$ corresponding to α (1.4).

It depends functorially on Y , namely

Lemma 6.2.2. *Let $Z \xrightarrow{f} Y$ be a K -morphism of smooth K -schemes and let $\alpha \in H^0(Y', \Omega_{Y'/K}^1)$, then $f'^* \phi_Y(\alpha) = \phi_Z((f'^*)^{ad} \alpha)$ where f'^* on the left (resp. on the right) is the pullback of classes in the Brauer group (resp. pullback of forms) by f' .*

Proof: By 6.2.1, $\phi_Z((f')^* \alpha) = [s_{(f')^*}^* (F_{Z/K} D_Z)]$ and $[s_{(f')^*}^* (F_{Z/K} D_Z)] = [(f'_d \circ Z \times_Y s_\alpha)^* (F_{Z/K} D_Z)]$ since $s_{(f')^*} = f'_d \circ Z \times_Y s_\alpha$ (1.4). Moreover by [7, 3.7], $[f'_d^* (F_{Z/K} D_Z)] = [f'_\pi^* (F_{Y/K} D_Y)]$ hence $[(f'_d \circ Z \times_Y s_\alpha)^* (F_{Z/K} D_Z)] = [(Z \times_Y s_\alpha)^* f'_\pi^* (F_{Y/K} D_Y)] = [(f'_\pi \circ Z \times_Y s_\alpha)^* (F_{Y/K} D_Y)] = [(s_\alpha \circ f')^* (F_{Y/K} D_Y)] = [f'^* s_\alpha^* (F_{Y/K} D_Y)] = f'^* [s_\alpha^* (F_{Y/K} D_Y)] = f'^* \phi_Y(\alpha)$ by 6.2.1 and using the equality $f'_\pi \circ Z \times_Y s_\alpha = s_\alpha \circ f'$.

Further diagram chasing through the cohomology long exact sequences gives control over the kernel of ϕ_Y , say Zariski locally,

Proposition 6.2.3. *Suppose further that Y is affine. Then there is an exact sequence (compatible with restriction to affine open subsets)*

$$\text{Pic}(Y) \rightarrow \text{coker} H^0(W^* - C_Y) \xrightarrow{\overline{\phi_Y}} \text{Br}(Y')_p \rightarrow 0.$$

Proof: It is a special case of [28, 1.7].

For $Y = T^*X$ where X is a smooth equidimensional K -scheme, ϕ_{T^*X} relates the canonical 1-form $\theta_{X'}$ on $T^*(X')$ to the class of the Azumaya algebra $F_{X/K} D_X$ (1.3) in $\text{Br}(T^*(X'))$. Indeed, note that the pullback of forms $W^* \Omega_{X/K}^1 \xrightarrow{W^*} \Omega_{X'/K}^1$ (1.5) is an isomorphism [23, 16.4.5] hence induces an isomorphism $(T^*X)' \rightarrow T^*(X')$, use it to identify $(T^*X)'$ and $T^*(X')$ and denote the resulting K -scheme T^*X' , then by [35, prop. 4.4 and 4.2] we have the

Proposition 6.2.4. $\phi_{T^*X}(\theta_{X'}) = [F_{X/K} D_X] \in \text{Br}(T^*X')$.

7 Lagrangianity

In this section, we complete the proof of the main theorem 2.2.1.

Namely, the first bound of theorem 5.3.2 allows one to construct a smooth compactification of an open dense subset of the p -support (prop. 7.2.1), "uniformly in p ", by reducing the problem to characteristic zero. From the description of the canonical form restricted to the p -support in terms of the p -curvature operator, given in section 6, and the analysis of the latter's action on the order of poles of differential forms (prop. 7.3.1), we get that the symplectic form has logarithmic poles along the boundary of the above compactification. We then conclude using Hodge theory in characteristic zero (7.4).

7.1 Poles and logarithmic poles

Let S be a scheme, let \overline{Y} be a smooth S -scheme and let D be a closed subscheme of \overline{Y} . The closed subscheme D is said to be a divisor with normal crossings relative to S if there is an étale covering $\mathcal{U} \xrightarrow{\pi} \overline{Y}$ and at each point $v \in \mathcal{U}$ local étale coordinates $\{u_1, \dots, u_{n_v}\} : V_v \rightarrow \mathbb{A}_S^{n_v}$ in which the closed subscheme $\pi^{-1}D := \mathcal{U} \times_{\overline{Y}} D$ is described by the equation $u_1 \dots u_{r_v} = 0$ for some $r_v \leq n_v$. The

notion of divisor with normal crossings relative to the base is stable under étale localization on \overline{Y} and base change. Note that the ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\overline{Y}}$ defining D is invertible and set for an $\mathcal{O}_{\overline{Y}}$ -module \mathcal{F} and $n \in \mathbb{Z}$, $\mathcal{F}(nD) := \mathcal{F} \otimes_{\mathcal{O}_{\overline{Y}}} \mathcal{I}^{\otimes \circ_{\overline{Y}}(-n)}$.

Then the inclusion $Y \xrightarrow{j} \overline{Y}$ of the open subscheme $\overline{Y} - D$ is affine and if m is a nonnegative integer there is a canonical embedding $\Omega_{\overline{Y}/S}^m(nD) \hookrightarrow j_*\Omega_{Y/S}^m$ sending $\eta \otimes t^{\otimes(-n)} \mapsto \eta|_Y/t^n$ where η and t are respectively a local section of $\Omega_{\overline{Y}/S}^m$ and a local equation of D . We use this embedding to view $\Omega_{\overline{Y}/S}^m(nD)$ as a subsheaf of $j_*\Omega_{Y/S}^m$. Note that by noetherianity and [21, 1.4.1 c1)], for every local section η of $j_*\Omega_{Y/S}^m$ there is an n such that $\eta \in \Omega_{\overline{Y}/S}^m(nD)$.

A local section of $j_*\Omega_{Y/S}^m$ which is in $\Omega_{\overline{Y}/S}^m(nD)$ is said to have poles of order at most n along D . One defines [16, II §3] a subcomplex $(\Omega_{\overline{Y}/S}^\bullet(\log D), d)$ of $j_*(\Omega_{Y/S}^\bullet, d_{Y/S})$ by the condition that a local section η of $j_*\Omega_{Y/S}^m$ belongs to $\Omega_{\overline{Y}/S}^m(\log D)$ if and only if η and $(j_*d_{Y/S})\eta$ have poles of order at most 1 along D . It is called the logarithmic de Rham complex of $D \subset \overline{Y}/S$ and a local section of $j_*\Omega_{Y/S}^m$ which is in $\Omega_{\overline{Y}/S}^m(\log D)$ is said to have logarithmic poles along D . Moreover in local étale coordinates $\{u_1, \dots, u_{n_v}\}$ in the neighborhood of a point $v \in \mathcal{U} \xrightarrow{\pi} \overline{Y}$ where as above π is an étale covering and $\pi^{-1}D := \mathcal{U} \times_{\overline{Y}} D$ is described by the equation $u_1 \dots u_{r_v} = 0$ for some $r_v \leq n_v$, $\Omega_{\overline{Y}/S}^1(\log D)$ is free of basis $\{du_1/u_1, \dots, du_{r_v}/u_{r_v}, du_{r_v+1}, \dots, du_{n_v}\}$, hence for all m , $\Omega_{\overline{Y}/S}^m(\log D) \cong \Lambda_{\mathcal{O}_{\overline{Y}}}^m \Omega_{\overline{Y}/S}^1(\log D)$ and the $\mathcal{O}_{\overline{Y}}$ -module $\Omega_{\overline{Y}/S}^m(\log D)$ is locally free of finite type.

7.2 Compactification of the p -supports

Let us fix coordinates on $\mathbb{A}_{\mathbb{Z}}^n = \text{spec}(\mathbb{Z}[x_1, \dots, x_n])$. For any scheme S they induce, compatibly with base change, coordinates on \mathbb{A}_S^n hence on $T^*(\mathbb{A}_S^n)$, as well as an open immersion $T^*(\mathbb{A}_S^n) \xrightarrow{r} \mathbb{P}_S^{2n}$ (by the Rees construction associated to the increasing filtration by the order of polynomials, see 5.2 and 5.3). Moreover if S is of positive characteristic, the choice of coordinates induces base change compatible identifications $\mathbb{A}_S^{n'} = \mathbb{A}_S^n$ and $T^*(\mathbb{A}_S^{n'}) = T^*(\mathbb{A}_S^n)$. Note also that if S is the spectrum of a field of positive characteristic, the open immersion $T^*(\mathbb{A}_S^{n'}) = T^*(\mathbb{A}_S^n) \xrightarrow{r} \mathbb{P}_S^{2n}$ matches that of 5.3.

Provided the above open immersions and identifications, here are some consequences of the first estimate of theorem 5.3.2.

Let S be an integral scheme dominant and of finite type over \mathbb{Z} and let M be a coherent left $D_{\mathbb{A}_S^n/S}$ -module. Suppose that the fiber of M at the generic point of S is a non zero holonomic left \mathcal{D} -module. Then by theorem 5.3.2 there is a dense open subset U of S such that for each closed point $u \in U$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$, the degree of the reduced closure $\overline{\{r(z)\}}^{\text{red}}$ of the image of $z \in T^*(\mathbb{A}_{k(u)}^{n'})$ by the open immersion $T^*(\mathbb{A}_{k(u)}^{n'}) = T^*(\mathbb{A}_{k(u)}^n) \xrightarrow{r} \mathbb{P}_{k(u)}^{2n}$ is bounded above independently of u (by $e(M)$ the \mathcal{D} -module multiplicity for the Bernstein filtration of the fiber of M at the

generic point of S). Hence since the $k(u)$ are perfect, by [23, 4.6.1, 4.2.8] and the invariance of the Hilbert polynomial under fields base change, [25, 2.4 and 2.1(b)] imply that the Hilbert polynomial of $\overline{\{r(z)\}}^{red}$ belongs to a finite set Φ , independent of u .

Let $\mathcal{H}_{e(M)} := \coprod_{P \in \Phi} \text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^{2n}}^P$ where $\text{Hilb}_{\mathbb{P}_{\mathbb{Z}}^{2n}}^P$ is the Hilbert scheme of $\mathbb{P}_{\mathbb{Z}}^{2n}$ of index P [25, p.17] and let $\mathcal{Z}_{e(M)} \subset \mathbb{P}_{\mathbb{Z}}^{2n} \times_{\text{spec}(\mathbb{Z})} \mathcal{H}_{e(M)}$ be the associated universal flat closed subscheme. Then by [25, 3.2] $\mathcal{H}_{e(M)}$ is projective over $\text{spec}(\mathbb{Z})$, in particular it is noetherian.

The first estimate of theorem 5.3.2 is used in the proof of the main theorem (2.2.1) through the following

Proposition 7.2.1. *Let T be a noetherian scheme and let $\mathcal{H}_{e(M),T} := \mathcal{H}_{e(M)} \times_{\text{spec} \mathbb{Z}} T$. Then there are a strictly positive integer N and a finite partition of $\mathcal{H}_{e(M),T}[1/N]$ into locally closed irreducible subsets \mathcal{S}_i such that if the \mathcal{S}_i are endowed with their reduced subschemes structure and $\mathcal{Z}_i := \mathcal{Z}_{e(M)} \times_{\mathcal{H}_{e(M)}} \mathcal{S}_i$, then for each i , the generic point of \mathcal{S}_i is of characteristic zero and there are a smooth open subset $\mathcal{U}_i \subset \mathcal{Z}_i^{red}$ surjecting onto the base \mathcal{S}_i and an open \mathcal{S}_i -immersion $\mathcal{U}_i \cap T^*(\mathbb{A}_{\mathcal{S}_i}^n) =: \mathcal{Y}_i \xrightarrow{j} \overline{\mathcal{Y}_i}$ into a smooth projective \mathcal{S}_i -scheme which is the complement of a divisor \mathcal{D}_i with normal crossings relative to \mathcal{S}_i .*

Moreover, for each i , let θ_i be the restriction of the canonical form $\theta_{\mathbb{A}_{\mathcal{S}_i}^n/\mathcal{S}_i}$ on $T^*(\mathbb{A}_{\mathcal{S}_i}^n)$ to \mathcal{Y}_i . Then the \mathcal{S}_i can also be chosen such that $d\theta_i$ has logarithmic poles along \mathcal{D}_i as soon as there is a fiber on which it has logarithmic poles and such that if $d\theta_i$ vanishes on the generic fiber of \mathcal{Y}_i then it vanishes on the whole of \mathcal{Y}_i .

Note that by noetherianity, there is a nonnegative integer m such that for each i the restriction θ_i of the canonical form $\theta_{\mathbb{A}_{\mathcal{S}_i}^n/\mathcal{S}_i}$ on $T^*(\mathbb{A}_{\mathcal{S}_i}^n)$ to \mathcal{Y}_i has poles of order at most m along \mathcal{D}_i .

Proof: Let us consider the analogous statement, with $\mathcal{H}_{e(M),T}$ replaced by one of its subschemes and note that its fulfillment for $(\mathcal{H}_{e(M),T})^{red}$ implies the proposition. Hence it is enough to prove the statement for all reduced closed subschemes of $\mathcal{H}_{e(M),T}$. Let \mathcal{S} be such a scheme. By noetherian induction on the set of reduced closed subschemes of $\mathcal{H}_{e(M),T}$ satisfying its conclusion, one may assume that the statement holds for all reduced closed subschemes of \mathcal{S} whose underlying space is $\neq \mathcal{S}$. There is a strictly positive integer N such that the generic points of the irreducible components of $\mathcal{S}[1/N]$ are of characteristic zero. If $\mathcal{S}[1/N]$ is empty then the statement holds. Suppose thus that $\mathcal{S}[1/N]$ is not empty. Let $z \in \mathcal{S}[1/N]$ be the generic point of an irreducible component and let \mathcal{Z}_z be the fiber of $(\mathcal{Z}_{e(M)} \times_{\mathcal{H}_{e(M)}} \mathcal{S}[1/N])^{red}$ at z . Then $\mathcal{Z}_z \subset \mathbb{P}_{k(z)}^{2n}$ is a scheme of finite type over $k(z)$, a field of characteristic zero, it is reduced by [23, 8.7.2 a)] and hence contains an open dense $k(z)$ -smooth subset $U \subset \mathcal{Z}_z$ by [23, 17.15.12]. By the resolution of singularities in characteristic zero, there is an open immersion $Y \xrightarrow{j} \overline{Y}$ of the quasi-projective variety $Y := U \cap T^*(\mathbb{A}_{k(z)}^n) \subset \mathbb{P}_{k(z)}^{2n}$ into a smooth projective scheme \overline{Y} over $k(z)$ which is the complement of a

divisor D with normal crossings relative to $k(z)$. Hence by [23, 8.10.5 and 17.7.8] there is an open affine neighborhood \mathcal{T} of z , which can be chosen, integral by [21, 2.1.9(ii)] since \mathcal{S} is reduced and such that there is a non-empty smooth open subset $\mathcal{U} \subset (\mathcal{Z}_{e(M)} \times_{\mathcal{H}_{e(M)}} \mathcal{T})^{red}$ surjecting onto \mathcal{T} as smooth morphisms are open and an open \mathcal{T} -immersion $\mathcal{U} \cap T^*(\mathbb{A}_{\mathcal{T}}^n) =: \mathcal{Y} \xrightarrow{j} \overline{\mathcal{Y}}$ into a smooth projective \mathcal{T} -scheme which is the complement of a divisor \mathcal{D} with normal crossings relative to \mathcal{T} .

Let θ be the restriction of the canonical form $\theta_{\mathbb{A}_{\mathcal{T}}^n/\mathcal{T}}$ on $T^*(\mathbb{A}_{\mathcal{T}}^n)$ to \mathcal{Y} . If $d\theta$ vanishes on the generic fiber of \mathcal{Y} then there is a dense open subset $V \subset \mathcal{T}$ on which $d\theta$ vanishes, $d\theta|_{\mathcal{Y}|_V} = 0$. If $d\theta$ does not vanish on the generic fiber of \mathcal{Y} , then set $V := \mathcal{T}$. There is also a dense open subset $W \subset V$ such that $d\theta|_{\mathcal{Y}|_W}$ has logarithmic poles along $\mathcal{D}|_W$ as soon as there is a fiber on which it has logarithmic poles. Indeed, since having logarithmic poles is an étale local condition, one may assume that $(\overline{\mathcal{Y}}|_V, \mathcal{D}|_V) = (\mathbb{A}_V^n = \text{spec}(V[y_1, \dots, y_n]), \{y_1 \dots y_r = 0\})$ for some $0 \leq r \leq n$ and one concludes by considering the vanishing loci in V of the remainders of divisions by the y_i 's.

By noetherian induction the statement holds for the reduced closed subscheme of \mathcal{S} whose underlying space is $\mathcal{S} - W$. Hence combining with the above on W , the statement holds for \mathcal{S} . This proves the proposition.

7.3 Action of the p -curvature operator on the order of poles

Let \overline{Y} be a smooth scheme over a field k of positive characteristic p , let D be a divisor with normal crossings relative to k (7.1) and let $Y \xrightarrow{j} \overline{Y}$ be the inclusion of the open subscheme $\overline{Y} - D$. Base changing by the Frobenius endomorphism of k , one sees that the closed subscheme $D' \subset \overline{Y}'$ is a divisor with normal crossings relative to k and that $Y' \xrightarrow{j'} \overline{Y}'$ is the open subscheme $\overline{Y}' - D'$. Suppose that Y is equidimensional and let $\mathcal{I}m(W^* - C_Y)$ be the image of the morphism $Z^1(F/k_* \Omega_{Y/k}^\bullet) \xrightarrow{W^* - C_Y} \Omega_{Y'/k}^1$ of abelian sheaves on $Y'(1.5)$.

Then there's the following inclusion of abelian subsheaves of $j'_* \Omega_{Y'/k}^2$

Proposition 7.3.1.

$$d(\Omega_{\overline{Y}'/k}^1((p-1)D') \cap j'_* \mathcal{I}m(W^* - C_Y)) \subset \Omega_{\overline{Y}'/k}^2(\log D')$$

Proof: Let η be a local section of $\Omega_{\overline{Y}'/k}^1((p-1)D') \cap j'_* \mathcal{I}m(W^* - C_Y) \subset j'_* \Omega_{Y'/k}^1$ and let $\mathcal{U} \xrightarrow{\pi} \overline{Y}$ be an étale covering as in the definition of a divisor with normal crossings (7.1). Since the pullback $\pi'^* \eta$ is a local section of

$$\Omega_{\mathcal{U}'/k}^1((p-1)\pi'^{-1}D') \cap j'_{\mathcal{U}*} \mathcal{I}m(W^* - C_{\mathcal{U}})$$

where $j_{\mathcal{U}}$ is the open immersion $\mathcal{U} - \pi^{-1}D \hookrightarrow \mathcal{U}$ and since being a section of $\Omega_{\overline{Y}'/k}^2(\log D')$ is an étale local condition, one may assume that at each point

$y \in \overline{Y}$ there are local étale coordinates $\{y_1, \dots, y_n\} : V_y \rightarrow \mathbb{A}_k^n$ in which the closed subscheme D is described by the equation $y_1 \dots y_r = 0$ for some $r \leq n$, n and r depending on y .

Let us prove that

$$\Omega_{\overline{Y}/k}^1((p-1)D') \cap j'_* \mathcal{I}m(W^* - C_Y) \subset j'_* B^1 \Omega_{Y'/k}^\bullet + \Omega_{\overline{Y}/k}^1(\log D')$$

where $B^1 \Omega_{Y'/k}^\bullet := \text{im}(\mathcal{O}_{Y'} \xrightarrow{d} \Omega_{Y'/k}^1)$ are the exact 1-forms. It implies the proposition. Let η be a local section of $j'_* \mathcal{I}m(W^* - C_Y)$, by lemma 7.3.2 below the canonical inclusion $\mathcal{I}m j'_*(W^* - C_Y) \hookrightarrow j'_* \mathcal{I}m(W^* - C_Y)$ is an isomorphism hence locally there is a section ζ of $j'_* Z^1(F_{/k*} \Omega_{Y/k}^\bullet)$ such that $\eta = j'_*(W^* - C_Y)\zeta$. Moreover if a local section ζ of $j'_* Z^1(F_{/k*} \Omega_{Y/k}^\bullet) \subset j'_* F_{/k*} \Omega_{Y/k}^1 = F_{/k*} j_* \Omega_{Y/k}^1$ does not belong to $F_{/k*} \Omega_{\overline{Y}/k}^1((p-1)D)$ then as C_Y is p^{-1} -linear and sends $Z^1(F_{/k*} \Omega_{\overline{Y}/k}^\bullet)$ to $\Omega_{\overline{Y}/k}^1$, $\eta = j'_*(W^* - C_Y)\zeta$ does not belong to $\Omega_{\overline{Y}/k}^1((p-1)D')$ [29, 0.2.5.4]. Note also that the étale coordinates $\{y_1, \dots, y_n\} : V_y \rightarrow \mathbb{A}_k^n$ above determine a splitting on $Y' \cap V'_y$ of the canonical short exact sequence

$$0 \rightarrow B^1(F_{/k*} \Omega_{Y/k}^\bullet) \rightarrow Z^1(F_{/k*} \Omega_{Y/k}^\bullet) \xrightarrow{C_Y} \Omega_{Y'/k}^1 \rightarrow 0$$

given in terms of local sections by $\Omega_{Y'/k}^1 \rightarrow Z^1(F_{/k*} \Omega_{Y/k}^\bullet) : \sum_{i=1}^n a_i dy'_i \mapsto \sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$. It induces a splitting on V'_y of the direct image j'_* of the above short exact sequence and hence a local section ζ of $j'_* Z^1(F_{/k*} \Omega_{Y/k}^\bullet)|_{V'_y}$ may uniquely be written as a sum $\zeta = \beta + \sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$ where β is a local section of $j'_* B^1(F_{/k*} \Omega_{Y/k}^\bullet)$ and the a_i 's are local sections of $j'_* \mathcal{O}_{Y'}$. Note finally that if ζ is a section of $F_{/k*} \Omega_{\overline{Y}/k}^1((p-1)D)$ then so is $\sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$ and that if $\sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$ is not a section of $F_{/k*} \Omega_{\overline{Y}/k}^1(1D)$ then it is not a section of $F_{/k*} \Omega_{\overline{Y}/k}^1((p-1)D)$ either. Indeed the proofs of both assertions reduce to $(\overline{Y}, D) = (\mathbb{A}_k^n = \text{spec}(k[y_1, \dots, y_n]), \{y_1 \dots y_r = 0\})$ with étale coordinates $\{y_1, \dots, y_n\}$ since the pullback by $\{y'_1, \dots, y'_n\} : V'_y \rightarrow \mathbb{A}_k^{n'}$ preserves the splitting and the order of poles. There the second assertion is a direct consequence of the factoriality of rings of polynomials with coefficients in a field while the first can be proved as follows.

Let $\zeta = dg + \sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$ be the above decomposition of a closed form on an affine open $\{f \neq 0\}$, for a rational function g and a polynomial f . Suppose that $\zeta \in F_{/k*} \Omega_{\overline{Y}/k}^1((p-1)D)$. Then $\sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i \in F_{/k*} \Omega_{\overline{Y}/k}^1((p-1)D)$. Indeed, multiplying by a high enough power of f^p , one may assume that ζ is a global section. Moreover by uniqueness of the decomposition and the corresponding splitting for forms without poles, one may also assume that $dg = \sum_{i=1}^n \partial_i(g) dy_i$ and $\sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$ are sections of $F_{/k*} \Omega_{\overline{Y}/k}^1(pD)$. Further using uniqueness of the decomposition and the splitting for forms without poles, and multiplying by $(y_1 \dots y_r)^p$, if $\sum_{i=1}^n F_{/k}^*(a_i) y_i^{p-1} dy_i$ was not a section of $F_{/k*} \Omega_{\overline{Y}/k}^1((p-1)D)$ then there would be an i such that

$y_1 \dots y_r$ divides $\partial_i(g) + F_{/k}^*(a_i)y_i^{p-1}$ but not $F_{/k}^*(a_i)y_i^{p-1}$, where g and a_i are polynomials. In particular there should be $1 \leq j \leq r$ such that $j \neq i$ and y_j divides $\partial_i(g) + F_{/k}^*(a_i)y_i^{p-1}$ but not $F_{/k}^*(a_i)y_i^{p-1}$. Expressing as polynomials in y_j and considering the degree zero terms would provide an equality $\partial_i(g_0) + F_{/k}^*((a_i)_0)y_i^{p-1} = 0$ with $(a_i)_0 \neq 0$. Since this cannot happen in characteristic p , the assertion holds.

Hence if $\zeta = \beta + \sum_{i=1}^{i=n} F_{/k}^*(a_i)y_i^{p-1}dy_i$ belongs to $F_{/k*}\Omega_{Y/k}^1((p-1)D)$ then $\sum_{i=1}^{i=n} F_{/k}^*(a_i)y_i^{p-1}dy_i$ is a section of $F_{/k*}\Omega_{Y/k}^1(1D) \cap j'_*Z^1(F_{/k*}\Omega_{Y/k}^\bullet) \subset Z^1(F_{/k*}\Omega_{Y/k}^\bullet(\log D))$. Thus

$$\begin{aligned} \eta &= j'_*(W^* - C_Y)\zeta = j'_*(W^* - C_Y)(\beta + \sum_{i=1}^{i=n} F_{/k}^*(a_i)y_i^{p-1}dy_i) \\ &= j'_*(W^*)\beta + j'_*(W^* - C_Y)(\sum_{i=1}^{i=n} F_{/k}^*(a_i)y_i^{p-1}dy_i) \end{aligned}$$

is a section of $j'_*B^1\Omega_{Y'/k}^\bullet + \Omega_{Y'/k}^1(\log D')$, as C_Y preserves forms with logarithmic poles [31, 7.2]. This concludes the proof of the proposition.

Lemma 7.3.2. *The canonical inclusion $\mathcal{I}m(j'_*(W^* - C_Y)) \hookrightarrow j'_*\mathcal{I}m(W^* - C_Y)$ is an isomorphism.*

Proof: The exact sequence of abelian sheaves on Y' (1.5)

$$0 \rightarrow \mathcal{O}_{Y'} \xrightarrow{F_{/k}^*} F_{/k*}\mathcal{O}_Y^* \xrightarrow{F_{/k*}d\log} Z^1(F_{/k*}\Omega_{Y/k}^\bullet) \xrightarrow{W^* - C_Y} \Omega_{Y'/k}^1$$

provides two short exact sequences

$$0 \rightarrow \text{coker}F_{/k}^* \xrightarrow{F_{/k*}d\log} Z^1(F_{/k*}\Omega_{Y/k}^\bullet) \xrightarrow{W^* - C_Y} \mathcal{I}m(W^* - C_Y) \rightarrow 0$$

and

$$0 \rightarrow \mathcal{O}_{Y'} \xrightarrow{F_{/k}^*} F_{/k*}\mathcal{O}_Y^* \rightarrow \text{coker}F_{/k}^* \rightarrow 0.$$

The associated long exact sequence to the first one

$$0 \rightarrow j'_*\text{coker}F_{/k}^* \xrightarrow{j'_*F_{/k*}d\log} j'_*Z^1(F_{/k*}\Omega_{Y/k}^\bullet) \xrightarrow{j'_*(W^* - C_Y)} j'_*\mathcal{I}m(W^* - C_Y) \rightarrow R^1j'_*\text{coker}F_{/k}^* \rightarrow \dots$$

shows that the lemma follows from the vanishing of $R^1j'_*\text{coker}F_{/k}^*$ which in turn by the long exact sequence associated to the second short exact sequence

$$\dots \rightarrow R^1j'_*\mathcal{O}_{Y'} \rightarrow R^1j'_*(F_{/k*}\mathcal{O}_Y^*) \rightarrow R^1j'_*\text{coker}F_{/k}^* \rightarrow R^2j'_*\mathcal{O}_{Y'} \rightarrow \dots$$

would follow from the vanishings of $R^1j'_*(F_{/k*}\mathcal{O}_Y^*)$ and $R^2j'_*\mathcal{O}_{Y'}$. Since the direct image $F_{/k*}$ preserves flasque sheaves and is exact as $F_{/k}$ is a homeomorphism, $R^qF_{/k*}(G) = 0$ for all abelian sheaves G and all $q > 0$ by [26, III 8.3] and $R^1j'_*(F_{/k*}\mathcal{O}_Y^*) \cong R^1(j'_* \circ F_{/k*})(\mathcal{O}_Y^*) = R^1(F_{/k*} \circ j'_*)(\mathcal{O}_Y^*) \cong F_{/k*}R^1j'_*(\mathcal{O}_Y^*)$. Hence by [23, 17.15.2 and 21.11.1] both $R^1j'_*(F_{/k*}\mathcal{O}_Y^*) \cong F_{/k*}R^1j'_*(\mathcal{O}_Y^*)$ and $R^2j'_*\mathcal{O}_{Y'}$ vanish by 7.3.3 below, thus proving the lemma.

Lemma 7.3.3. *Let $U \xrightarrow{j} Y$ be an open immersion. Suppose that Y is a locally factorial noetherian scheme. Then $R^q j_*(\mathcal{O}_U^*) = 0$ for all $q > 0$.*

Proof: By [26, III 8.1] $R^q j_*(\mathcal{O}_U^*)$ is the abelian sheaf associated to the presheaf $V \mapsto H^q(U \cap V, \mathcal{O}_{U \cap V}^*)$, V open in Y . Since by 6.1.5 $H^q(U \cap V, \mathcal{O}_{U \cap V}^*) = 0$ for all $q \geq 2$, $R^q j_*(\mathcal{O}_U^*) = 0$ for all $q \geq 2$.

For $q = 1$, $R^1 j_*(\mathcal{O}_U^*)$ is the abelian sheaf associated to the presheaf $V \mapsto H^1(U \cap V, \mathcal{O}_{U \cap V}^*) \cong \text{Pic}(U \cap V)$, V open in Y . Let $\mathcal{L}_{U \cap V} \in \text{Pic}(U \cap V)$. By [23, 21.6.11] it extends to an invertible sheaf $\mathcal{L}_V \in \text{Pic}(V)$ hence $\mathcal{L}_{U \cap V}$ is trivial on the trace of an open covering of V trivializing \mathcal{L}_V and in particular the corresponding section of the associated sheaf $R^1 j_*(\mathcal{O}_U^*)$ is 0. Thus $R^1 j_*(\mathcal{O}_U^*) = 0$, finishing the proof.

7.4 Conclusion

Recall the statement 2.2.1

Theorem. *Let S be an integral scheme dominant and of finite type over \mathbb{Z} , let X be a smooth S -scheme of relative dimension n and let M be a coherent left $D_{X/S}$ -module. Suppose that the fiber of M at the generic point of S is a holonomic left \mathcal{D} -module. Then there is a dense open subset U of S such that the p -support of the fiber of M at each closed point u of U is a lagrangian subscheme of $(T^*(X'_u), \omega_{X'_u})$.*

Proof: By the remark 2.4.3, we may assume that the fiber of M at the generic point of S is non zero. Hence by theorem 3.1.1, there is a dense open subset U_a of S such that the p -support of the fiber of M at each closed point u of U_a is equidimensional of dimension n . Since by 4.2 one may further suppose that $X/S = \mathbb{A}_S^n/S$, theorem 6.1.3 implies that there is a dense open subset U_b of U_a such that for each closed point $u \in U_b$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$ the Azumaya algebra $F_{\mathbb{A}^n/k(u)*} D_{\mathbb{A}_{k(u)}^n}$ on $T^*(\mathbb{A}_{k(u)}^{n'})$ splits on $(\overline{\{z\}}^{red})^{reg}$. Therefore, $k(u)$ being perfect (2.2), by 6.2.4, 6.2.2 and 6.2.3 the restriction of the canonical form $\theta_{\mathbb{A}_{k(u)}^{n'}}$ to each of the $(\overline{\{z\}}^{red})^{reg}$ is a section of $\mathcal{I}m(W^* - C_{(\overline{\{z\}}^{red})^{reg}})$, where we identified $(\overline{\{z\}}^{red})^{reg}$ and $(\overline{\{z\}}^{red})^{reg'}$ by perfection of $k(u)$.

Moreover, by 7.2 and proposition 7.2.1, there is a dense open subset U_c of U_b such that for each closed point $u \in U_c$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$, there are a smooth open dense subscheme Y_z of $(\overline{\{z\}}^{red})^{reg}$ and an open immersion $Y_z \xrightarrow{j} \overline{Y}_z$ into a smooth projective scheme, which is the complement of a divisor D_z with normal crossings relative to $k(u)$. In addition there is a nonnegative integer m , independent of u , such that the restriction of the canonical form to Y_z has poles of order at most m along D_z . Hence inverting all primes $\leq m$, one may suppose that $m \leq \text{char}(k(u)) - 1$ and

thus by proposition 7.3.1, $\omega_{\mathbb{A}_{k(u)}^{n'}}|_{Y_z} := d\theta_{\mathbb{A}_{k(u)}^{n'}}|_{Y_z}$ has logarithmic poles along D_z .

Actually by proposition 7.2.1, there is a finite set Ξ such that for each $i \in \Xi$ there are an integral scheme \mathcal{S}_i whose generic point is of characteristic zero, a smooth \mathcal{S}_i -scheme \mathcal{Y}_i , an open immersion $\mathcal{Y}_i \xrightarrow{j} \overline{\mathcal{Y}_i}$ into a smooth projective \mathcal{S}_i -scheme which is the complement of a divisor \mathcal{D}_i with normal crossings relative to \mathcal{S}_i and a relative 1-form $\theta_i \in \Omega_{\overline{\mathcal{Y}_i}/\mathcal{S}_i}^1(m\mathcal{D}_i)$ such that for each closed point $u \in U_c$ and each z generic point of an irreducible component of $p\text{-supp}(M_u)$, there is $i(z) \in \Xi$ such that $Y_z \xrightarrow{j} \overline{Y_z}$ and $\theta_{\mathbb{A}_{k(u)}^{n'}}|_{Y_z}$ are deduced from $\mathcal{Y}_{i(z)} \xrightarrow{j} \overline{\mathcal{Y}_{i(z)}}$ and $\theta_{i(z)}$, base changing by a $k(u)$ -point of $\mathcal{S}_{i(z)}$. Moreover by construction of the \mathcal{S}_i 's (7.2.1), if $d\theta_{\mathbb{A}_{k(u)}^{n'}}|_{Y_z} \in \Omega_{Y_z/k}^2(\log D_z)$ then $d\theta_{i(z)} \in \Omega_{\overline{\mathcal{Y}_i}/\mathcal{S}_i}^2(\log \mathcal{D}_i)$, hence in particular $d\theta_0 \in \Omega_{\mathcal{Y}_0}^2(\log D_0)$, where here and below, the subscript 0 denotes restriction to the generic fiber. As the generic fiber is over a field of characteristic zero, the canonical inclusion $\Omega_{\mathcal{Y}_0}^\bullet(\log D_0) \subset (j_0)_* \Omega_{\mathcal{Y}_0}^\bullet$ is a quasi-isomorphism [17, 3.1.8], implying that the class of $d\theta_0$ in the hypercohomology of the logarithmic de Rham complex is zero. Hence $d\theta_0$ vanishes by the degeneracy at E_1 of the logarithmic Hodge to de Rham spectral sequence [17, 3.2.13 (ii) and 3.2.14] and so, by construction of the \mathcal{S}_i 's (7.2.1), the symplectic form $\omega_{\mathbb{A}_{k(u)}^{n'}} := d\theta_{\mathbb{A}_{k(u)}^{n'}}$

vanishes on the open dense subset Y_z of $(\overline{\{z\}})^{red, reg}$. Thus by the above there is a dense open subset U of $U_c \subset U_a$ such that for each closed point u of U , the symplectic form vanishes on a dense open subset of $p\text{-supp}(M_u)$. Since by definition of U_a , $p\text{-supp}(M_u)$ is equidimensional of dimension n for all closed points $u \in U_a$, this concludes the proof of the theorem.

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